On The Minimum-Weight Forward (Weakly) Fundamental Cycle Basis Problem in Directed Graphs

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Abstract

The cycle space of a directed graph is generated by a cycle basis, where, in general, cycles are allowed to have both forward and backward arcs. In a forward cycle, all arcs have to follow the given direction. We study the existence, structure, and computational complexity of minimum-weight forward cycle bases in directed graphs. We give a complete structural characterization of digraphs that admit weakly fundamental (and hence integral) forward cycle bases, showing that this holds if and only if every block is either strongly connected or a single arc. We further provide an easily verifiable characterization of when a strongly connected digraph admits a forward fundamental cycle basis, proving that such a basis exists if and only if the set of directed cycles has cardinality equal to the cycle rank; in this case, the basis is unique and computable in polynomial time, and nonexistence can likewise be certified efficiently.

Lastly, we show that while minimum-weight forward fundamental cycle bases can be found in polynomial time whenever they exist, the minimum-weight forward weakly fundamental cycle basis problem is NP-hard via a polynomial-time reduction from the minimum-weight weakly fundamental cycle basis problem on digraphs with metric weights.

Keywords

cycle bases, forward cycle bases, fundamental cycle bases, weakly fundamental cycle bases, minimum cycle basis, complexity

1 Introduction

Cycle bases provide a compact representation of all cycles in a graph and play a key role in optimization problems such as periodic scheduling, electrical network analysis, and modeling of chemical and biological pathways (see, e.g., Kavitha et al. [3]). Different optimization contexts call for different classes of cycle bases. For instance, aiming at railway timetable optimization, Nachtigall [8] introduced a mixed-integer programming formulation of the *periodic event scheduling problem* (PESP) based on fundamental cycle bases, which was later extended to integral cycle bases by Liebchen and Peters [4] and to forward cycle bases by Lindner et al. [5] and Masing et al. [7]. The choice of cycle basis can significantly influence computational efficiency, motivating the study of minimum-weight cycle basis problems for each class of cycle bases.

While existence and complexity results are well established for many classes of cycle bases [1]. An exception is the computational complexity of the minimum-weight integral cycle basis problem, which remains open. Forward cycle bases are less well understood. In [2], Gleiss, Leydold, and Stadler show that a digraph admits a forward cycle basis if and only if each block is either strongly connected or a single arc, and that a minimum-weight forward cycle basis can be computed in polynomial time. Masing et al. [7] show that forward cycle bases can improve the efficiency of branch-and-cut in integral cycle basis formulations of PESP. They also prove that any graph admits an orientation with a strongly fundamental forward cycle basis, and that a forward integral cycle basis always exists for so-called line-based event-activity networks constructed from a public transport line plan.

Several open questions remain regarding forward cycle bases, including the identification of structural graph properties that guarantee the existence of integral, weakly fundamental, and strongly fundamental forward cycle bases, the computational complexity of deciding their existence, and the problem of computing a minimum-weight forward cycle basis within a given class. From an applied perspective, when a forward cycle basis does not exist, a related extremal problem arises: finding a cycle basis that maximizes the number of forward cycles.

The remainder of the paper is organized as follows. In Section 2, we introduce basic notions and definitions related to directed graphs, cycle spaces, and various classes of cycle bases. Section 3 presents results on the existence of forward cycle bases and characterizations of weakly fundamental and fundamental forward cycle bases. In Section 4, we establish the NP-hardness of the minimum-weight forward weakly fundamental cycle basis problem through a polynomial-time reduction. Finally, Section 5 summarizes our contributions and outlines open questions for future research.

2 Preliminaries and basic notions

Let D be a finite directed graph (digraph), where $V(D)$ is the vertex set and $A(D) \subseteq V(D) \times V(D)$ is the arc set. An *(undirected) walk* in D is a sequence $P = v_0 e_1 v_1 e_2 \dots e_k v_k$ such that for each $i = 1, \dots, k$ the arc e_i is either $(v_{i-1}, v_i) \in A(D)$ or $(v_i, v_{i-1}) \in A(D)$ (i.e. the arc may be traversed in either direction). If every $e_i = (v_{i-1}, v_i) \in A(D)$ we say P is a *directed walk*. The vertices v_0 and v_k are the *start* and *end* of P, denoted start(P) and end(P). If end(P) = start(Q), we denote by PQ the *concatenation* of walks P and Q. A *(directed) path* is a (directed) walk with all vertices distinct. A *(directed) circuit* is a closed (directed) walk (i.e. start(P) = end(P)), and a *(directed) cycle* is a circuit in which all vertices except the first and last are distinct. Whenever convenient, we identify a path with the ordered sequence of its arcs or vertices, rather than with its formal representation. A digraph is *strongly connected* if every ordered pair of vertices is joined by a directed path, and *weakly connected* if there is a (possibly non-directed) path between every pair of vertices.

For a forest $\mathcal{T} \subset D$, i.e., a subgraph that does not contain any cycle, we write

$$\mathcal{T}^* := A(D) \setminus A(\mathcal{T})$$

for the set of chords (arcs not in $\mathcal{T}$). If $V(\mathcal{T}) = V(D)$, then $\mathcal{T}$ is *spanning*. A weakly connected forest is a *tree*.

INOC '26, Liège (Belgium)

10.48786/inoc.2026.06

If $\mathscr{K}$ is a field, $C \in \mathscr{K}^{A(D)}$ and $e \in A(D)$, then $C(e)$ denotes the coordinate of C corresponding to the arc e. The *support* of a vector $C \in \mathscr{K}^{A(D)}$ is denoted

$$\operatorname{supp}(C) := \{e \in A(D) : C(e) \neq 0\}.$$

We often identify the vector C with the set $\operatorname{supp}(C)$ and write "$e \in C$" to mean $e \in \operatorname{supp}(C)$ (equivalently $C(e) \neq 0$). In particular, we interpret a cycle C in D as a vector in $\mathscr{K}^{A(D)}$ that we also call C as follows: If $C = v_0 e_1 v_1 e_2 \dots e_k v_k$ with $v_0 = v_k$, the *incidence vector* is given by

$$C(e) := \begin{cases} 1 & \text{if } e = e_i \text{ for some } i \in \{1, \dots, k\} \text{ and } e_i = (v_{i-1}, v_i), \\ -1 & \text{if } e = e_i \text{ for some } i \in \{1, \dots, k\} \text{ and } e_i = (v_i, v_{i-1}), \\ 0 & \text{otherwise.} \end{cases}$$

Definition 1 (cycle space). The *cycle space* of D over a field $\mathscr{K}$ is the subspace of $\mathscr{K}^{A(D)}$ defined by:

$$\mathscr{C}_{\mathscr{K}}(D) := \left\{ C \in \mathscr{K}^{A(D)} \;\middle|\; \sum_{e \in \delta^+(v)} C(e) = \sum_{e \in \delta^-(v)} C(e) \right\}. \quad (1)$$

The dimension of the cycle space of D is called *cycle rank* (cyclomatic number, cyclotomic number) and is denoted by $\mu(D)$. By means of its incidence vector, any cycle in D can be seen as an element of the cycle space.

Definition 2 (classes of cycle bases). A *cycle basis* $\mathscr{B}$ of D is a basis of $\mathscr{C}_{\mathscr{K}}(D)$ composed by incidence vectors of cycles in D. Moreover, we call $\mathscr{B}$:

(1) *undirected*: if $\mathscr{B}$ is a basis of $\mathscr{C}_{\mathbb{F}_2}(D)$, where $\mathbb{F}_2$ is the finite field with two elements.
(2) *integral*: if any cycle C of D can be written as an integer linear combination of vectors in $\mathscr{B}$.
(3) *weakly fundamental* (Whitney [11]) : if there exists an ordering of vectors in $\mathscr{B} = \{C_1, \dots, C_\mu\}$ such that for all $1 \leq i \leq \mu$ there exists $e \in C_i$ such that $e \notin \bigcup_{j<i} C_j$.
(4) *(strictly) fundamental:* if there exists a spanning forest $\mathscr{T}$ in D such that $\mathscr{B} = \{C_{\mathscr{T}}^{(e)} \mid e \in \mathscr{T}^*\}$, where $C_{\mathscr{T}}^{(e)}$ is the only cycle contained in $\mathscr{T} \cup \{e\}$.

For an in-depth introduction to cycle spaces and bases, we refer to [3] of which we adopt the same notation. Definition 2 in fact provides a hierarchy: Any fundamental cycle basis is weakly fundamental, any weakly fundamental cycle basis is integral, and any integral cycle basis is undirected [3].

For $C \in \mathscr{C}_{\mathbb{R}}(D)$ and $e \in \operatorname{supp}(C)$, we can either have $C(e) > 0$ ("e is forward") or $C(e) < 0$ ("e is backward"). We write $C \geq \mathbf{0}$ if $C(e) > 0$ for all $e \in \operatorname{supp}(C)$, i.e., if C contains only forward arcs. In particular, this holds for any directed cycle. We further define

$$\mathscr{F}(D) := \{C \mid C \text{ is a directed cycle in } D\}.$$

Definition 3 (forward cycle basis). A cycle basis $\mathscr{B}$ of $\mathscr{C}_D(\mathbb{R})$ is a *forward cycle basis* if all cycles C in $\mathscr{B}$ are directed.

We prefer the term *forward* here, since "directed cycle basis" is used in [3] for an arbitrary cycle basis of $\mathscr{C}_D(\mathbb{R})$, although the latter need not be composed of directed cycles. In the same way as arbitrary cycle bases, forward cycle bases can be categorized as undirected, integral, weakly fundamental, or fundamental.

Lastly, the notion of separability is essential to characterize digraphs which admit a forward cycle basis:

Definition 4 (separable digraph). A digraph D is *separable* if there exists a vertex $v \in V(D)$ such that the digraph $D \setminus \{v\}$ is not weakly connected.

If a graph is separable, its maximal non-separable subgraphs form a partition and are called *blocks* of D. We briefly note that the cycle space of a digraph naturally decomposes into the direct sum of the cycle spaces of its blocks:

Observation 5. *Let* $\mathbf{B}$ *be the set of blocks of a digraph* D*. Then, for any field* $\mathscr{K}$*,*

$$\mathscr{C}_{\mathscr{K}}(D) = \bigoplus_{B \in \mathbf{B}} \mathscr{C}_{\mathscr{K}}(B).$$

In particular, an (undirected/integral/weakly fundamental/fundamental) cycle basis for D can be constructed by assembling (undirected/integral/weakly fundamental/fundamental) cycle bases for each block.

To answer the question regarding the existence of forward weakly fundamental and integral cycle bases, we will use the concept of ear decomposition of a directed graph. A (directed) ear $P \subset D$ of a subgraph $\overline{D}$ of D is a (directed) path or cycle in D such that only the first and last vertex of P are in $\overline{D}$ while all the interior vertices of P are not in $\overline{D}$.

Definition 6 ((directed) ear decomposition). A (directed) ear decomposition of a digraph D with at least two vertices is a sequence of nested subgraphs $D_1 \subset \dots \subset D_t = D$ such that

(1) D_1 is a (directed) cycle,
(2) $D_i = D_{i-1} \cup P_i$, where P_i is a (directed) ear of D_i for $1 < i \leq t$.

It is a known result in graph theory that a digraph admits a directed ear decomposition if and only if it is strongly connected.

Proposition 7 ([1]). *A strongly connected digraph* D *admits a directed ear decomposition* $D_1 \subset \dots \subset D_t = D$*. Furthermore,* $t = \mu(D) = |A(D))| - |V(D)| + 1$ *and the ear decomposition can be computed in* $\mathcal{O}(\min(|V(D)|, \mu(D)) \cdot (|V(D)| + |A(D)|))$ *time.*

3 Results on the existence of forward cycle bases

In this section, we answer questions regarding the existence of integral, weakly fundamental, and strictly fundamental forward cycle bases.

Proposition 8. *Let* D *be a digraph. The following are equivalent:*

(1) D *admits a forward weakly fundamental cycle basis.*
(2) D *admits a forward integral cycle basis.*
(3) D *admits a forward undirected cycle basis.*
(4) D *admits a forward cycle basis.*
(5) *Each block of* D *is strongly connected or a single arc.*

Proof. The equivalence of (4) and (5) is Theorem 7 in [2]. The implications (1) ⇒ (2) ⇒ (3) ⇒ (4) are clear. Now assume that (5) holds. By Observation 5, if there exists a weakly fundamental cycle basis for each block of D, a forward weakly fundamental cycle basis can be naturally obtained for D. Therefore, we only need to show that a weakly fundamental cycle basis exists for a block B of D. The case of a single arc is trivial. Otherwise, B is strongly connected. Therefore it admits a directed ear decomposition $D_1 \subset \dots \subset D_t$ by Proposition 7.

From this ear decomposition, we can construct a weakly fundamental cycle basis as follows: Let $C_1 := D_1$. For $i > 1$, consider the ear P_i and let $v, v' \in V(P_i)$ be the first and last vertex of P_i. Since D_{i-1} is strongly connected, there exists a directed walk P_i' from v' to v in D_{i-1}. Since $V(P_i') \subset D_{i-1}$ and P_i is an ear of D_{i-1}, the only vertices in $V(P_i) \cap V(P_i')$ are v and v', thus $C_i := P_i P_i' \subset D_i$

is a directed cycle. The set $\mathscr{B} := \{C_i\}_{i=1}^t$ is a weakly fundamental cycle basis of B, since $t = |A(B)| - |V(B)| + 1$, and for all $i > 1$, there is an edge $e \in A(P_i) \subset A(D_i) \setminus A(D_{i-1})$ that is not contained in $\cup_{j=1}^{i-1} C_j \subset D_{i-1}$. □

In particular, every strongly connected digraph D has a forward weakly fundamental cycle basis, as it admits a directed ear decomposition (Proposition 7).

Similarly, when not all blocks of D are either a single arc or strongly connected, we can consider the following extremal cycle basis problem [7]: Find a cycle basis such that the number of directed cycles is maximum. We obtain:

Proposition 9. *Let D be a digraph, let $\mathscr{B}$ be a (undirected, integral, or weakly fundamental) cycle basis of D with a maximum number of directed cycles, and let $\mathscr{D}$ be the set of strongly connected components of D. Then*

$$|\mathscr{B} \cap \mathscr{F}(D)| = \sum_{\overline{D} \in \mathscr{D}} \mu(\overline{D}). \tag{2}$$

Proof. Any directed cycle in D is contained in some strongly connected component of D. In each component $\overline{D} \in \mathscr{D}$, we can have at most $\mu(\overline{D})$ linearly independent cycles, so that for any cycle basis $\mathscr{B}$ of D holds $|\mathscr{B} \cap \mathscr{F}(D)| \leq \sum_{\overline{D} \in \mathscr{D}} \mu(\overline{D})$.

We will now construct a weakly fundamental cycle basis $\mathscr{B}$ of D such that (2) holds. At first, we choose a spanning tree of each strongly connected component of D. We then connect these trees to a spanning forest of D and consider the corresponding fundamental cycle basis $\mathscr{B}'$ of D. Any strongly connected graph admits a forward cycle basis and hence a forward weakly fundamental cycle basis by Proposition 8. In $\mathscr{B}'$, for each strongly connected component $\overline{D} \in \mathscr{D}$, we consider the cycles that arise from chords in $\overline{D}$. These constitute a fundamental cycle basis of $\overline{D}$, which we replace by a forward weakly fundamental cycle basis of $\overline{D}$. This way, we can transform $\mathscr{B}'$ to a weakly fundamental cycle basis $\mathscr{B}$ of D that contains precisely $\sum_{\overline{D} \in \mathscr{D}} \mu(\overline{D})$ directed cycles. □

A different proof for Proposition 8 and 9, using a notion of cycle subspaces and a contraction argument, is given by Masing [6].

As noted in [7], not all digraphs admit a forward fundamental cycle basis. We now show an easily verifiable characterization of digraphs which admit a forward fundamental cycle basis.

We will make use of the following lemma that states that every element of the cycle space with forward arcs only lives in the cone generated by the directed cycles.

Lemma 10 ([1]). *Let D be a directed graph and let $\overline{C} \in \mathscr{C}_{\mathbb{R}}(D)$ with $\overline{C} \geq \mathbf{0}$. Then there exist nonnegative scalars $\{\lambda_C\}_{C \in \mathscr{F}(D)}$ such that*

$$\overline{C} = \sum_{C \in \mathscr{F}(D)} \lambda_C C.$$

Moreover, if $\overline{C}$ has integer coordinates, then the coefficients λ_C can be chosen to be integers.

We can now give our surprisingly simple characterization.

Theorem 11. *Let D be a digraph such that each block is strongly connected or a single arc. Then D admits a forward fundamental cycle basis if and only if*

$$|\mathscr{F}(D)| = \mu(D).$$

In this case, $\mathscr{F}(D)$ constitutes the unique forward cycle basis of D.

Proof. Assume that a forward fundamental cycle basis $\mathscr{B}$ exists. Let $\mathscr{T} \subset D$ be the spanning forest that induces the basis, and let $\mathscr{T}^*$ denote the set of chords of $\mathscr{T}$. Then the cycle basis is

$$\mathscr{B} = \{C_{\mathscr{T}}^{(e)} \mid e \in \mathscr{T}^*\},$$

where $C_{\mathscr{T}}^{(e)}$ denotes the unique cycle contained in $\mathscr{T} \cup \{e\}$. Note that for all $C \in \mathscr{F}(D)$ and all $e \in A(D)$, we have $C(e) = 1$ if and only if $e \in C$ and $C(e) = 0$ otherwise. Since $\mathscr{B} \subset \mathscr{F}(D)$ and $|\mathscr{B}| = \mu(D)$, it suffices to show $\mathscr{F}(D) \subset \mathscr{B}$.

Let $\overline{C} \in \mathscr{F}(D)$. As $\{C_{\mathscr{T}}^{(e)}\}_{e \in \mathscr{T}^*}$ is an integral cycle basis, there exist integers $\{\lambda_e\}_{e \in \mathscr{T}^*}$ with

$$\overline{C} = \sum_{e \in \mathscr{T}^*} \lambda_e C_{\mathscr{T}}^{(e)}.$$

For each chord $\bar{e} \in \mathscr{T}^*$ the arc $\bar{e}$ belongs only to the cycle $C_{\mathscr{T}}^{(\bar{e})}$ among the cycles in $\mathscr{B}$, and moreover $C_{\mathscr{T}}^{(\bar{e})}(\bar{e}) = 1$ because $C_{\mathscr{T}}^{(\bar{e})}$ is directed. Comparing the value of both sides on the arc $\bar{e}$ we obtain

$$\overline{C}(\bar{e}) = \sum_{e \in \mathscr{T}^*} \lambda_e C_{\mathscr{T}}^{(e)}(\bar{e}) = \lambda_{\bar{e}}.$$

Thus $\lambda_{\bar{e}} = 1$ if $\bar{e} \in \overline{C}$ and $\lambda_{\bar{e}} = 0$ otherwise. Hence $\overline{C}$ is the sum of precisely those $C_{\mathscr{T}}^{(\bar{e})}$ that correspond to the chords contained in $\overline{C}$. If more than one such $C_{\mathscr{T}}^{(\bar{e})}$ appeared in the sum, the support of the sum would be a union of two or more arc-disjoint cycles. This contradicts that $\overline{C}$ is a cycle. Therefore exactly one coefficient is 1 and the others are 0, so $\overline{C} = C_{\mathscr{T}}^{(\bar{e})}$ for some $\bar{e} \in \mathscr{T}^*$. Consequently $\overline{C} \in \mathscr{B}$, proving $\mathscr{F}(D) \subset \mathscr{B}$ and hence $\mathscr{B} = \mathscr{F}(D)$.

Conversely, suppose $|\mathscr{F}(D)| = \mu(D)$, i.e., the family of directed cycles has a size equal to the cycle rank. Since by Proposition 8, a forward cycle basis $\mathscr{B}$ exists and is contained in $\mathscr{F}(D)$, it must be equal to $\mathscr{F}(D)$. Given $\overline{C} \in \mathscr{B}$ we define, for each $e \in \overline{C}$, the set

$$\mathscr{H}(e) := \{C \in \mathscr{B} \setminus \{\overline{C}\} \mid e \in C\}.$$

Using the standard characterization of fundamental cycle bases [10], a cycle basis $\mathscr{B}$ is fundamental if and only if for every $\overline{C} \in \mathscr{B}$ there exists an arc $\bar{e} \in \overline{C}$ such that

$$\bar{e} \notin \bigcup_{C \in \mathscr{B} \setminus \{\overline{C}\}} C,$$

or equivalently, such that $|\mathscr{H}(\bar{e})| = 0$. Let $\bar{e} \in \arg\min_{e \in \overline{C}} (|\mathscr{H}(e)|)$. Consider the vector

$$\hat{C} := \sum_{C \in \mathscr{B} \setminus \{\overline{C}\}} C - \overline{C} \in \mathscr{C}_{\mathbb{R}}(D). \tag{3}$$

We prove that $\hat{C}$ is non-negative. For every $e \notin \overline{C}$, we clearly have $\hat{C}(e) \geq 0$. Suppose now that $e \in \overline{C}$. Then

$$\hat{C}(e) = \sum_{C \in \mathscr{H}(e)} C(e) - |\mathscr{H}(\bar{e})|\overline{C}(e) = |\mathscr{H}(e)| - |\mathscr{H}(\bar{e})|\overline{C}(e) \geq 0$$

since $|\mathscr{H}(\bar{e})|$ was chosen to be minimum and $\overline{C}(e) \in \{0, 1\}$ as $\mathscr{B}$ is forward. Then, by Lemma 10, there exists $\lambda_C \geq 0$ for all $C \in \mathscr{F}(D)$ such that

$$\hat{C} = \sum_{C \in \mathscr{F}(D)} \lambda_C C. \tag{4}$$

Since $\mathscr{F}(D) = \mathscr{B}$, and $\mathscr{B}$ is a basis of $\mathscr{C}_{\mathbb{R}}(D)$, the linear combinations in equation (3) and (4) have the same coefficients. In particular:

$$-|\mathscr{H}(\bar{e})| = \lambda_{\overline{C}} \geq 0,$$

which implies $|\mathscr{H}(\bar{e})| = 0$ as wanted. □

Theorem 11 gives a polynomial-time algorithm at hand to decide whether a digraph D admits a forward fundamental cycle basis: Check at first whether all blocks of D are strongly connected or a single arc. If not, then D cannot have a forward cycle basis by Proposition 8. Otherwise, compute a forward weakly fundamental cycle basis $\mathscr{B}$ using a directed ear decomposition. We may then verify that for all $\overline{C} \in \mathscr{B}$ exists $e \in \overline{C}$ such $e \notin \bigcup_{C \in \mathscr{B} \setminus \{\overline{C}\}} C$ to check whether $\mathscr{B}$ is also strictly fundamental. If yes, then we have found a forward fundamental cycle basis. If no, then D cannot have a forward fundamental cycle basis, as the existence of such a basis would contradict the uniqueness of a forward cycle basis by Theorem 11.

4 NP-hardness of the Minimum-Weight Forward Weakly Fundamental Cycle Basis Problem

In this section, we consider the problem of finding forward cycle bases with a minimum weight. More precisely, let D be a digraph endowed with arc weights $w : A(D) \to \mathbb{R}_{\geq 0}$. For each cycle C in D, we define its weight as

$$w(C) := \sum_{e \in C} w(e),$$

and the total weight of a cycle basis $\mathscr{B}$ is given by

$$w(\mathscr{B}) := \sum_{C \in \mathscr{B}} w(C).$$

Definition 12. Given a digraph D and arc weights $w : A(D) \to \mathbb{R}_{\geq 0}$, the *minimum-weight forward (weakly fundamental, fundamental) cycle basis problem* is to find a forward (weakly fundamental, fundamental) cycle basis of $\mathscr{B}$ such that $w(\mathscr{B})$ is minimum or to decide that no forward cycle basis exists.

The existence of forward (weakly fundamental) cycle bases has already been characterized by Proposition 8. The problem is polynomial-time solvable for arbitrary forward cycle bases [2]. For forward fundamental cycle bases, we can use Theorem 11.

Corollary 13. *A minimum–weight forward fundamental cycle basis of D, if it exists, can be computed in polynomial time. If no such basis exists, its nonexistence can also be certified in polynomial time.*

Proof. If a forward fundamental cycle basis exists, then it is unique by Theorem 11, and hence trivially optimal. We can decide the existence as discussed at the end of §3. □

The goal of the remainder of this section is to show that the minimum-weight forward weakly fundamental cycle basis problem is NP-hard. We give a polynomial-time reduction from the non-forward version on digraphs with metric weights, whose NP-hardness – even for uniform weights – has been established by Rizzi [9].

Let D be a digraph and let further $w : A(D) \to \mathbb{R}_{\geq 0}$ be *metric* arc weights, i.e., for any arc $e = (u, v)$, $w(e)$ is the cost of a shortest u-v-path w.r.t. w. We construct a directed graph $\overleftrightarrow{D}$ by adding to $A(D)$, for each $e = (u, v) \in A(D)$, the arc $\overleftarrow{e} := (v, u)$, with weight equal to $w((u, v))$. This transformation clearly produces a directed graph of size polynomial in $|V(D)| + |A(D)|$, where each weakly connected component is strongly connected. Define

$$\mathrm{OPT}^{\mathrm{wF}}(D) := \min \left\{ w(\mathscr{B}) \;\middle|\; \begin{array}{c} \mathscr{B} \text{ is a weakly fundamental} \\ \text{cycle basis of } D \end{array} \right\}$$

and

$$\mathrm{OPT}^{\mathrm{wF,fwd}}(\overleftrightarrow{D}) := \min \left\{ w(\mathscr{B}) \;\middle|\; \begin{array}{c} \mathscr{B} \text{ is a weakly fundamental} \\ \text{forward cycle basis of } \overleftrightarrow{D} \end{array} \right\}.$$

Observation 14. *The cycle-rank of $\overleftrightarrow{D}$ is equal to*

$$\overleftrightarrow{\mu} := 2|A(D)| - |V(D)| + 1.$$

In particular, the cycle-rank of $\overleftrightarrow{D}$ exceeds that of D by exactly $|A(D)|$. We now focus on a distinguished family of directed cycles in $\overleftrightarrow{D}$. For each arc $e = (u, v) \in A(D)$, let $S^{(e)}$ denote the directed 2-cycle in $\overleftrightarrow{D}$ formed by the arcs (u, v) and (v, u). Since w is metric, $S^{(e)}$ is a smallest weight cycle in $\overleftrightarrow{D}$ containing e. These cycles constitute precisely the set of all 2-cycles of $\overleftrightarrow{D}$. We denote this set by

$$\mathscr{S} := \{S^{(e)} \mid e \in A(D)\}.$$

Furthermore, for every cycle C in D, we define a directed counterpart $\overleftrightarrow{C}$ by choosing, for each arc $e = (u, v) \in C$, one of the two orientations (u, v) or (v, u) such that $\overleftrightarrow{C}$ is a directed cycle in $\overleftrightarrow{D}$. We call $\overleftrightarrow{C}$ a *directed lift* of C in $\overleftrightarrow{D}$. There are exactly two such directed lifts, corresponding to the two traversal directions of C; either choice is admissible, and we fix one. In particular, for every $e \in C$, there exists $e^{(C)} \in \{e, \overleftarrow{e}\}$ such that $e^{(C)} \in \overleftrightarrow{C}$. See Figure 1 for an illustrative example.

Finally, there is a natural extension of any cycle basis $\mathscr{B}$ of D to a *forward* cycle basis of $\overleftrightarrow{D}$, obtained by augmenting the set $\{\overleftrightarrow{C} \mid C \in \mathscr{B}\}$ with the 2-cycles $S^{(e)}$:

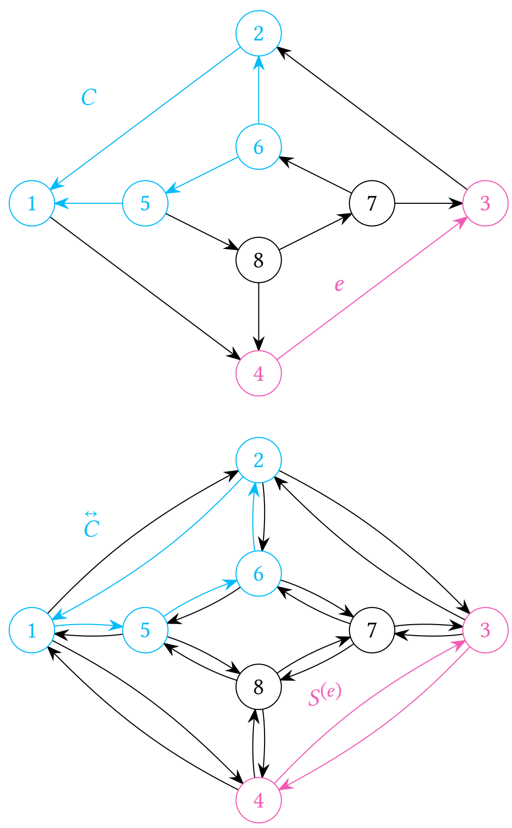

Figure 1: Reduction applied to the Wagner graph with highlighted 2-cycle, and cycle lifting.

Lemma 15 (Forward lifting cycle bases). *If $\mathscr{B}$ is a cycle basis of D, then*

$$\overleftrightarrow{\mathscr{B}} := \{\overleftrightarrow{C} \mid C \in \mathscr{B}\} \cup \mathcal{S} \tag{5}$$

is a forward cycle basis of $\overleftrightarrow{D}$.

Proof. Since $|\overleftrightarrow{\mathscr{B}}| = 2|A(D)| - |V(D)| + 1 = \overleftrightarrow{\mu}$ (Observation 14), it suffices to show that $\overleftrightarrow{\mathscr{B}}$ spans the cycle space $\mathscr{C}_{\mathbb{R}}(\overleftrightarrow{D})$. Observe that for every cycle C in $\overleftrightarrow{D}$, the vector

$$C' := C - \sum_{\substack{e \in A(D) \\ \text{s.t. } C(\overleftarrow{e}) \neq 0}} S^{(e)} \tag{6}$$

has support in $A(D)$, as this corresponds to substituting arc $\overleftarrow{e}$ in C with e whenever $\overleftarrow{e}$ is not in $A(D)$ by subtracting the cycle $S^{(e)}$ from C. In particular, $C' \in \mathrm{Span}(\mathscr{B})$, so that $C \in \mathrm{Span}(\mathscr{B} \cup \mathcal{S})$. It now suffices to show that each cycle $C \in \mathscr{B}$ lies in $\mathrm{Span}(\overleftrightarrow{\mathscr{B}} \cup \mathcal{S})$. However, in this case $\left(\overleftrightarrow{C}\right)' = C$, and (6) realizes C as a linear combination of $\overleftrightarrow{C} \in \overleftrightarrow{\mathscr{B}}$ and elements of $\mathcal{S}$. □

Lemma 16 (Weakly fundamental lifting and weights). *Let $\mathscr{B}$ be any weakly fundamental cycle basis of D. Then $\overleftrightarrow{\mathscr{B}}$ is a forward weakly fundamental cycle basis of $\overleftrightarrow{D}$ and*

$$w(\overleftrightarrow{\mathscr{B}}) \;=\; w(\mathscr{B}) \;+ \sum_{e \in A(D)} 2\,w(e). \tag{7}$$

In particular, lifting increases the objective by an additive constant independent of the choice of $\mathscr{B}$ and

$$\mathrm{OPT}^{\mathrm{wF,fwd}}(\overleftrightarrow{D}) \leq \mathrm{OPT}^{\mathrm{wF}}(D) + \sum_{e \in A(D)} 2\,w(e). \tag{8}$$

Proof. Since B is weakly fundamental, there exists an ordering of the cycles $\mathscr{B} = \{C_i\}_{i=1}^{\mu}$, where $\mu := \mu(D)$, such that for every $1 < k \leq \mu$ there is an arc $e_k \in C_k$ with $e_k \notin \bigcup_{i<k} C_i$. Then also $e_k^{(C_k)} \in \overleftrightarrow{C}_k$ and $e_k^{(C_k)} \notin \bigcup_{i<k} \overleftrightarrow{C}_i$.

An ordering that makes $\overleftrightarrow{\mathscr{B}}$ weakly fundamental is therefore obtained by first listing the cycles in $\mathcal{S} \setminus \{S^{(e_k)} \mid k = 1, \dots, \mu\}$, then $\overleftrightarrow{C}_1$, and subsequently, for each $k = 1, \dots, \mu$, placing $\overleftrightarrow{C}_k$ immediately before $S^{(e_k)}$. Since exactly one of e_k or $\overleftarrow{e}_k$ belongs to $\overleftrightarrow{C}_k$, while both are contained in $S^{(e_k)}$, this ordering ensures that $\overleftrightarrow{\mathscr{B}}$ is indeed weakly fundamental.

Verifying (7) and (8) is straightforward. □

To complete the reduction, we must prove the converse: There is an optimal forward weakly fundamental cycle basis of $\overleftrightarrow{D}$ that is obtained as a lifting of an optimal weakly fundamental cycle basis of D. First, we prove that we can easily construct an optimal forward weakly fundamental cycle basis of D that contains the cycles in $\mathcal{S}$:

Lemma 17 (Existence of minimum-weight forward weakly fundamental cycle basis containing $\mathcal{S}$). *There exists a minimum-weight forward weakly fundamental cycle basis $\mathscr{B}^*$ of $\overleftrightarrow{D}$ such that $\mathcal{S} \subset \mathscr{B}^*$.*

Proof. Let $\mathscr{B}$ be a minimum-weight forward weakly fundamental cycle basis of $\overleftrightarrow{D}$. We order the basis $\mathscr{B} = \{C_k\}_{k=1}^{\overleftrightarrow{\mu}}$ such that for every index $1 < k \leq \overleftrightarrow{\mu}$ there exists an arc $e_k \in C_k$ with $e_k \notin \bigcup_{i<k} C_i$. Removing the chord set $\{e_k\}_{k=1}^{\overleftrightarrow{\mu}}$ from $\overleftrightarrow{D}$ yields a spanning forest $\mathcal{T}$: If we remove $e_{\overleftrightarrow{\mu}}$ from $\overleftrightarrow{D}$, all weakly connected components remain connected, the cycle rank of the resulting graph drops by exactly one, and $\mathscr{B} \setminus \{e_{\overleftrightarrow{\mu}}\}$ is weakly fundamental cycle basis. Iterating this process, we ultimately arrive at a spanning forest [9].

By construction each cycle C_k is contained in $\mathcal{T} \cup \{e_i\}_{i \leq k}$. Let

$$\mathfrak{C}_k := \{\overline{C} \in \mathcal{F}(D) \mid \overline{C} \subset \mathcal{T} \cup \{e_i\}_{i \leq k} \text{ and } e_k \in \overline{C}\}$$

be the family of all directed cycles that lie inside $\mathcal{T} \cup \{e_i \mid i \leq k\}$ and contain e_k. A minimum-weight forward weakly fundamental basis can be obtained by choosing, for each k, any directed cycle of minimum weight in $\mathfrak{C}_k$, in particular C_k, is a minimum-weight directed cycle in $\mathfrak{C}_k$.

For each $e \in A(D)$ we proceed as follows: Consider the corresponding 2-cycle $S^{(e)}$ in $\overleftrightarrow{D}$. Since $\mathcal{T}$ is a spanning tree of $\overleftrightarrow{D}$, it cannot contain both e and its reverse $\overleftarrow{e}$. Consequently, at least one of these two opposite arcs coincides with a chord e_j. Let $k \geq j$ be the largest index such that e_k is one of the two arcs of $S^{(e)}$.

By construction, we have

$$S^{(e)} \subseteq \mathcal{T} \cup \{e_i\}_{i \leq k} \quad \text{and} \quad e_k \in S^{(e)},$$

which implies $S^{(e)} \in \mathfrak{C}_k$. Moreover, $S^{(e)}$ is a minimum-weight cycle in $\mathfrak{C}_k$, and therefore

$$w(S^{(e)}) = w(C_k).$$

We now construct a new cycle basis $\mathscr{B}^*$ by replacing C_k with $S^{(e)}$ in $\mathscr{B}$ for each such 2-cycle in $\overleftrightarrow{D}$. The resulting basis is a minimum-weight forward weakly fundamental cycle basis satisfying $\mathcal{S} \subset \mathscr{B}^*$, as claimed. □

Lemma 18 (Projection to the original problem). *Let $\mathscr{B}^*$ be an optimal forward weakly fundamental basis of $\overleftrightarrow{D}$ that contains $\mathcal{S}$. Then, removing the family $\mathcal{S}$ from $\mathscr{B}^*$ and projecting each remaining directed cycle to its underlying cycle in D yields a minimum weakly fundamental cycle basis $\mathscr{B}$ of D. Moreover,*

$$w(\mathscr{B}^*) = w(\mathscr{B}) + \sum_{e \in A(D)} 2\,w(e), \tag{9}$$

Proof. Consider $\widetilde{\mathscr{B}} := \mathscr{B}^* \setminus \mathcal{S}$ and define for each $\tilde{C} \in \widetilde{\mathscr{B}}$ the projection $\mathrm{proj}(\tilde{C}) := C \in \mathscr{C}_{\mathbb{R}}(D)$ by setting, for every $e \in A(D)$,

$$C(e) := \tilde{C}(e) \;-\; \tilde{C}(\overleftarrow{e}).$$

Because $\mathscr{B}^*$ is weakly fundamental in $\overleftrightarrow{D}$ there exists an ordering of its cycles $\{\tilde{C}_k\}_{k=1}^{\tilde{\mu}}$ in which each cycle contains an arc $\tilde{e}_k$ not used by any earlier cycle. Defining $C_i := \mathrm{proj}(\tilde{C}_i)$, this induces an ordering of $\mathscr{B} := \{C_i\}_{i=1}^{\mu}$.

We claim that for every k, the arc $e_k \in \{\tilde{e}_k, \overleftrightarrow{\tilde{e}_k}\}$ with $e_k \in C_k$ does not belong to any cycle C_i with $i < k$. Indeed, by construction, $\tilde{e}_k$ is not contained in any cycle preceding $\tilde{C}_k$ in the original ordering of $\mathscr{B}^*$; hence no earlier $\tilde{C}_i$ contains the same directed arc. In particular, the cycle $S^{(e_k)}$ appears after $\tilde{C}_k$ in the ordering. But then also any cycle in $\mathscr{B}^*$ containing $\overleftrightarrow{\tilde{e}_k}$ can only appear later than $\tilde{C}_k$. We conclude that $\mathscr{B}$ is weakly fundamental in D, as required.

Equation (9) follows from the fact that the weights of the cycles in $\mathscr{B}$ is equal to the weight of the cycles in $\widetilde{\mathscr{B}}$ and that $w(\mathcal{S}) = \sum_{e \in A(D)} 2\,w(e)$. Then (9) implies that

$$\mathrm{OPT}^{\mathrm{wF,fwd}}(\overleftrightarrow{D}) \geq \mathrm{OPT}^{\mathrm{wF}}(D) + \sum_{e \in A(D)} 2\,w(e) \tag{10}$$

and together with Lemma 16, this implies that the inequality is actually an equality. Therefore, $\mathscr{B}$ is a minimum-weight weakly fundamental cycle basis of D. □

Theorem 19 (NP-hardness). *The minimum-weight forward weakly fundamental cycle basis problem is NP-hard. The NP-hardness persists when the weights are metric or uniform.*

Proof. Given an instance (D, w) of the minimum-weight weakly fundamental cycle basis problem with uniform weights, we construct $(\overleftrightarrow{D}, \widetilde{w})$ with uniform and hence metric weights in polynomial time. By Lemma 16, every feasible weakly fundamental cycle basis of (D, w) lifts to a feasible forward weakly fundamental cycle basis of $(\overleftrightarrow{D}, \widetilde{w})$, with objective value increased by an instance-dependent constant. Conversely, by Lemma 18, any minimum-weight forward weakly fundamental cycle basis of $(\overleftrightarrow{D}, \widetilde{w})$ projects to an optimal weakly fundamental cycle basis of (D, w).

Therefore, a polynomial-time algorithm for the minimum forward weakly fundamental cycle basis problem would imply a polynomial-time algorithm for the minimum weakly fundamental cycle basis problem, which is NP-hard for uniform weights [9]. □

For completeness, we finally show that the assumption that (D, w) has metric weights is necessary for our reduction to work. We present a counterexample for which Lemma 17 and Lemma 18 do not hold.

Example 20. Consider the digraph D with $V(D) = \{a, b, c\}$ and $A(D) = \{(a, b), (b, c), (c, a)\}$ with weights $w((a, b)) = w((b, c)) := 1$ and $w((c, a)) = 3$. In D there exists a unique cycle C; in particular, $\mathcal{B} := \{C\}$ forms the minimum-weight weakly fundamental cycle basis of D.

However, $\overleftrightarrow{\mathcal{B}} = \{\overleftrightarrow{C}\} \cup \mathcal{S}$ is not a minimum-weight forward weakly fundamental cycle basis of $\overleftrightarrow{D}$, since it has weight 15, while the forward weakly fundamental cycle basis

$$\begin{aligned} \mathcal{B}' &:= \{S^{((a,b))}, S^{((b,c))}, C', C''\}, \quad \text{where} \\ C' &= a, (a, b), b, (b, c), c, (c, a), a \quad \text{and} \\ C'' &= a, (a, c), c, (c, b), b, (b, a), a \end{aligned}$$

is optimal and has weight 14. Thus, Lemma 18 does not hold for non-metric weights, as $\overleftrightarrow{\mathcal{B}}$ is not an optimal forward weakly fundamental cycle basis of $\overleftrightarrow{D}$. Moreover, Lemma 17 does not hold, as any forward cycle basis of $\overleftrightarrow{D}$ containing all three cycles in $\mathcal{S}$ must have weight at least 15.

5 Conclusion

We studied the existence and structural properties of forward cycle bases, focusing on weakly and strictly fundamental variants. Our results clarify their relation to classical cycle basis classes and contribute to a better theoretical understanding of forward cycle bases in directed graphs.

Several questions remain open. In particular, given that the minimum-weight weakly fundamental cycle basis problem is APX-hard, it is natural to ask if there is a similar result for the forward case. Moreover, the computational complexity of the minimum (forward) integral cycle basis problem is unknown.

References

[1] Jørgen Bang-Jensen and Gregory Z. Gutin. 2009. *Digraphs: Theory, Algorithms and Applications* (2 ed.). Springer London. XXII + 795 pages. doi:10.1007/978-1-84800-998-1 Second Edition.

[2] Petra Gleiss, Josef Leydold, and Peter Stadler. 2003. Circuit bases of strongly connected digraphs. *Discussiones Mathematicae Graph Theory* 23 (01 2003), 241. doi:10.7151/dmgt.1200

[3] Telikepalli Kavitha, Christian Liebchen, Kurt Mehlhorn, Dimitrios Michail, Romeo Rizzi, Torsten Ueckerdt, and Katharina A. Zweig. 2009. Cycle bases in graphs characterization, algorithms, complexity, and applications. *Computer Science Review* 3, 4 (2009), 199–243. doi:10.1016/j.cosrev.2009.08.001

[4] Christian Liebchen and Leon Peeters. 2009. Integral cycle bases for cyclic timetabling. *Discrete Optimization* 6, 1 (2009), 98–109. doi:10.1016/j.disopt.2008.09.003

[5] Niels Lindner, Christian Liebchen, and Berenike Masing. 2021. Forward Cycle Bases and Periodic Timetabling. In *21st Symposium on Algorithmic Approaches for Transportation Modelling, Optimization, and Systems (ATMOS 2021) (Open Access Series in Informatics (OASIcs), Vol. 96)*, Matthias Müller-Hannemann and Federico Perea (Eds.). Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl, Germany, 2:1–2:14. doi:10.4230/OASIcs.ATMOS.2021.2

[6] Berenike Masing. [n. d.]. *Enriched Periodic Timetabling.* Ph. D. Dissertation. Freie Universität Berlin. Submitted (2025).

[7] Berenike Masing, Niels Lindner, and Patricia Ebert. 2023. Forward and Line-Based Cycle Bases for Periodic Timetabling. *Operations Research Forum* 4, 3 (June 2023), 53. doi:10.1007/s43069-023-00229-0

[8] Karl Nachtigall. 1996. *Cutting planes for a polyhedron associated with a periodic network.* Technical Report 112-96/17. Deutsches Zentrum für Luft- und Raumfahrt.

[9] Romeo Rizzi. 2009. Minimum Weakly Fundamental Cycle Bases Are Hard To Find. *Algorithmica* 53, 3 (2009), 402–424. doi:10.1007/s00453-007-9112-8

[10] Maciej Syslo. 2006. On cycle bases of a graph. *Networks* 9 (10 2006), 123 – 132. doi:10.1002/net.3230090203

[11] Hassler Whitney. 1935. On the Abstract Properties of Linear Dependence. *American Journal of Mathematics* 57, 3 (1935), 509–533. http://www.jstor.org/stable/2371182