Putting Tutte's counterexample to Tait's conjecture in perspective to Hamiltonicity and non-Hamiltonicity in certain planar cubic graphs

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Abstract

Tait's conjecture, which states that every (polyhedral) 3-connected planar cubic graph has a Hamiltonian cycle, was disproved by Tutte in the last century. This subsequently raised the challenge of finding a smallest counterexample, a problem ultimately resolved by Holton and McKay. Using the graphs of prisms and Tutte Fragments, we construct an infinite partial order of Hamiltonian and non-Hamiltonian graphs in which Tutte's counterexample to Tait's conjecture appears, in a certain sense, as a minimal element. We further observe that the minimum-cardinality counterexamples of Holton and McKay to Tait's conjecture (as well as certain generalizations of these) are also contained in this partial order.

Keywords

Hamiltonian cycles, cubic graphs, polyhedral graphs, Tutte fragment

1 Introduction and preliminary considerations

The *Tutte Fragment* (TF) is a particular planar subcubic graph with only three 2-valent vertices [6]. Figure 1 illustrates it by placing the three 2-valent vertices a, b, and c at the vertices of an equilateral triangle. In 1946, W. T. Tutte [6] constructed a 3-connected planar cubic graph on 46 vertices admitting no Hamiltonian cycle by suitably linking copies of the TF via their 2-valent vertices. This counterexample disproved for the first time the longstanding conjecture of Tait from 1884, claiming that every 3-regular, 3-connected, planar graph is Hamiltonian. To obtain this insight, specific properties of the TF are exploited; see Lemma 1.1 which is provable by exhaustion.

Lemma 1.1 ([6] Hamiltonian paths of the TF). *Let TF be the graph displayed in Figure 1 with correspondingly assigned labels a, b, and c for the 2-valent vertices. Then, no Hamiltonian path of the TF with endpoints a and b exists. However, two Hamiltonian paths with endpoint-pair (a, c), and four Hamiltonian paths with endpoint-pair (b, c) exist.*

Note, however, that D. W. Barnette proposed in 1969 the conjecture that every planar, 3-connected, cubic bipartite graph is Hamiltonian. So far, there is only one major partial solution to this conjecture; see [1]. Calling such graphs *Barnette graphs*, it was shown there that if two of the color classes of a Barnette graph consist of quadrilaterals and hexagons only, then it is Hamiltonian (in fact, the main result there is even somewhat more general). Moreover, it was shown in [2] that if Barnette's conjecture is false, then the decision problem of whether a Barnette graph is Hamiltonian, is NP-complete. The paper [7] studies decompositions into Hamiltonian cycles of prisms over 3-connected planar bipartite cubic graphs; furthermore, it shows that prisms over any 3-connected cubic graph are Hamiltonian. Prior to [7], it had been proved by Fleischner, without invoking the Four Color Theorem, that the prism of a 2-connected planar cubic graphs is Hamiltonian [4].

The aim of the current work is to show that Tutte's counterexample, as well as the counterexample of Holton and McKay to Tait's conjecture are not isolated incidences. Rather, they are, in a sense, minimal examples in an infinite family of counterexamples to Tait's conjecture. Moreover, the same TF-inflations (see below for this concept) applied to even-sided prisms yield Hamiltonian graphs (Theorem 2.4 and Corollary 2.5).

In our considerations all graphs are assumed to be simple and undirected. For standard terminology of graph theory we refer to the textbook by Bondy and Murty [3]. The *n-prism* $C_n \square K_2$ is the Cartesian product of C_n, the cycle of length $n \geq 3$, and K_2, the path on two vertices. For simplicity, assume the following classification of the vertices: Let the vertex set be partitioned into vertices of a *base cycle* $\{s_i : i \in \mathbb{Z}_n\}$ and vertices of a *top cycle* $\{t_i : i \in \mathbb{Z}_n\}$; accordingly, the set of edges is given by the union of both cycles' edges $\{s_i s_{i+1}, t_i t_{i+1} : i \in \mathbb{Z}_n\}$ and the set of *pillars* $\{s_i t_i : i \in \mathbb{Z}_n\}$.

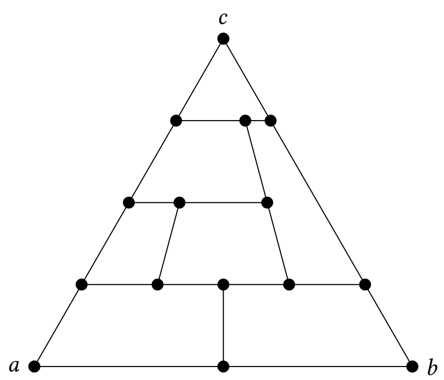

Figure 1: The celebrated Tutte fragment (TF) found in 1946. No Hamiltonian path of the TF with endpoints a and b exists. However, certain Hamiltonian paths with endpoint-pair (a, c) as well as (b, c) exist.

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A *triangle inflation* (see also [7, p. 51]) replaces a 3-valent vertex v with a triangle Δ having vertex set $V(\Delta) = \{v^a, v^b, v^c\}$ such that each of the triangle's vertices is connected to a unique former neighbor of v. Just as one speaks of inflating a 3-valent vertex into a triangle, we will speak of *TF-inflations*.

Definition 1.2 (TF-inflation). A TF-inflation of a 3-valent vertex v is a triangle inflation of v followed by the following operations: The edges of Δ are deleted, and each v^x, $x \in \{a, b, c\}$, is identified with a unique corresponding 2-valent vertex x of the TF. The resulting graph has 14 additional vertices and is said to derive from the original graph by TF-inflation. If the original graph containing v is 3-regular (and 3-connected), then the graph resulting from the TF-inflation is as well.

Given the asymmetry of the TF, depending on which of the former v-incident edges are associated to v^a, v^b, or v^c of the TF, we obtain six possibilities for a TF-inflation. However, in the case of prisms, for every TF-inflation we insist that $v^c = c$ is incident to the corresponding pillar, leaving two possibilities for a TF-inflation in n-prisms. After TF-inflations we regard all edges of the prism contained in $\{s_i t_i, s_i^c t_i, t_i^c s_i, t_i^c s_i^c : i \in \mathbb{Z}_n\}$ as *pillars of the TF-inflated prism.* TF-inflations on a prism are illustrated in Figure 2.

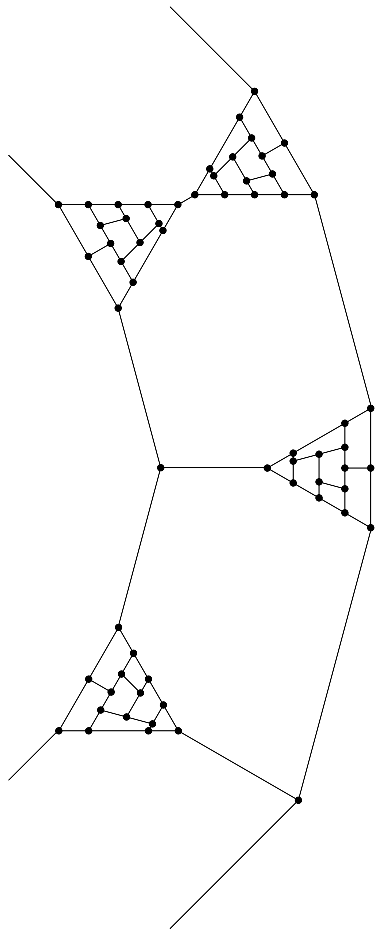

Figure 2: Three pillars of an n-prism with single- and double-sided TF-inflations: A single TF-inflation at the pillar's top (respectively bottom) vertex is shown for the lowermost (respectively middle) pillar. In the case of the uppermost pillar, both of its vertices have been subjected to a TF-inflation.

2 Results

We start with some preparatory observations.

Lemma 2.1. *The number $r \in \mathbb{N}$ of pillars contained in a Hamiltonian cycle of $C_n \square K_2$ can only be $r \in \{2, n\}$. If $r = 2$, the two pillars must belong to the same quadrangle of the prism.*

Proof. Requiring a cycle to include fewer than two pillars does not result in a Hamiltonian cycle. If more than two but fewer than all pillars are contained in a Hamiltonian cycle H of $C_n \square K_2$, one can find distinct pillars p, p' and p'' such that, firstly, all pillars $u_1, \ldots, u_k$ (with $k \geq 1$) between p and p' are uncovered by H; and, secondly, a nonnegative number of uncovered pillars $u'_1, \ldots, u'_{k'}$ ($k' \geq 0$) lies between p' and p''; see Figure 3. Without loss of generality, we can assume that the part of H being incident to $u_1, \ldots, u_k$ passes through the base cycle of $C_n \square K_2$. Consequently, the top vertices of $u_1, \ldots, u_k$ would necessarily remain uncovered by H, contradicting Hamiltonicity.

The argument applies analogously to the case where exactly two distinct pillars are covered by H, but do not belong to the same quadrangle of the prism. □

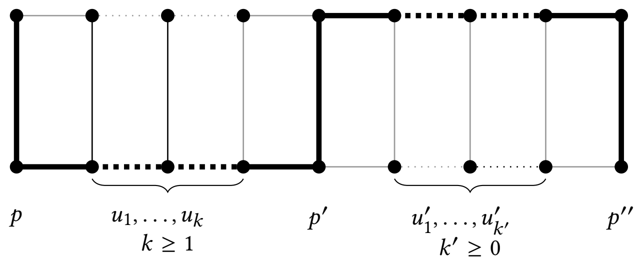

Figure 3: Each Hamiltonian cycle of $C_n \square K_2$ either covers exactly two consecutive pillars or all pillars. Pillars that are uncovered in a given Hamiltonian cycle are labeled u_i, u'_i.

Definition 2.2. A Hamiltonian cycle H in a prism $C_n \square K_2$ is called *meandering* if no two edges of the top or bottom cycle are adjacent in H.

It follows that a meandering Hamiltonian cycle in $C_n \square K_2$ must contain all pillars of $C_n \square K_2$.

Lemma 2.3. *The following two assertions hold.*

(i) If n is odd, there is no Hamiltonian cycle of $C_n \square K_2$ containing all pillars.
(ii) If n is even, there are precisely two Hamiltonian cycles covering all pillars of $C_n \square K_2$; both are meandering.

Proof. Let B and T denote the edge sets of the base cycle and the top cycle of $C_n \square K_2$, respectively. Let Π denote the set of pillars of $C_n \square K_2$. Let $C = (e_1, \ldots, e_{2_n})$ denote the sequence of edges of a Hamiltonian cycle of $C_n \square K_2$ further assuming that $e_1 = s_0 t_0$. There are only two scenarios meeting our assumptions: either $(e_{2n}, e_2) \in B \times T$, or $(e_{2n}, e_2) \in T \times B$. Both scenarios propagate a meandering form $C \in \Pi \times T \times \Pi \times B \times \cdots \times \Pi \times T \times \Pi \times B \times \cdots$ — with swapped roles of B and T in case of the initial assumption $(e_{2n}, e_2) \in T \times B$. Only when n is even, such a meandering Hamiltonian cycle can successfully be closed; according to the two possible initial assumptions, two distinct meandering Hamiltonian cycles then exist. □

Remark 1. *In contrast, it is easy to verify that a Hamiltonian cycle of $C_n \square K_2$ containing only two pillars shared by a quadrangle exist for arbitrary n: It consists of the two pillars and all edges of the top and bottom cycles except the two edges joining the pillars in $C_n \square K_2$; this Hamiltonian cycle does not meander as in the other cases.*

Observation 1. *Due to the structure of the TF, recall Lemma 1.1, whenever a vertex of $C_n \square K_2$ has been TF-inflated, each Hamiltonian cycle of the inflated version of $C_n \square K_2$ must contain the incident pillar.*

Theorem 2.4. *Suppose G is obtained from $C_n \square K_2$ by applying a TF-inflation to at least one vertex of $r \geq 3$ distinct pillars of $C_n \square K_2$. Then, the following assertions hold.*

(i) A graph G is Hamiltonian if and only if n is even. Furthermore, for even n, every Hamiltonian cycle of G naturally corresponds to one of the two meandering Hamiltonian cycles of $C_n \square K_2$.
(ii) If n is odd, then G has Hamiltonian paths.

Proof. Let us first prove (i). By contrapositive, if n is odd and G has a Hamiltonian cycle H, then the cycle H' resulting from the contraction of those edges in H that stem from a TF-inflation, would yield a Hamiltonian cycle for $C_n \square K_2$. From the assumption combined with Observation 1 we get that H' necessarily covers at least three pillars of $C_n \square K_2$, implying by Lemma 2.1 that all pillars must be covered by H'. However, since n is odd, this yields a contradiction to Lemma 2.3. Consequently, no Hamiltonian cycle H in G exists.

Let us now prove that for even n, Hamiltonicity of G ensues. By Lemma 2.3 (ii), there is a Hamiltonian cycle H covering all pillars of $C_n \square K_2$. Whenever a TF-inflation is applied to a vertex v of $C_n \square K_2$, we can augment H towards a feasible Hamiltonian cycle for the post-inflation version of $C_n \square K_2$: Without loss of generality, we can assume $v = s_i$, $s_i s_{i+1} \in H$, and adjacency of s_i^b and s_{i+1}. Then one simply replaces the former pillar $s_i t_i$ by the union of the new pillar $s_i^c t_i$ and the Hamiltonian path connecting s_i^c with s_i^b (running within the TF and existing due to Lemma 1.1). As these inter-TF Hamiltonian paths are not unique (see Lemma 1.1), multiple versions of the augmented Hamiltonian cycle are conceivable; see Section 2.1. However, they all derive from the same Hamiltonian cycle in $C_n \square K_2$. The argument applies for iterated TF-inflation; consequently, G is Hamiltonian.

Let us show that an existing Hamiltonian path H of G for even n is—after contracting its inter-TF edges—coincident with a meandering path of $C_n \square K_2$. The number of pillars covered by H in G is invariant under inter-TF paths contractions. By assumption, at least three pillars of G are covered by H, implying that at least three pillars in $C_n \square K_2$ are covered by the contracted cycle H'. By Lemma 2.1, this implies that all n pillars must be covered by H', which in turn means, by Lemma 2.3 (ii), that H' is a meandering Hamiltonian cycle.

Next, let us show (ii). Insert into G an additional pillar which is not subject to any TF-inflation. This augmented graph equivalently derives from the even-sided prism $C_{n+1} \square K_2$ by suitable TF-inflations and therefore possesses a Hamiltonian cycle H_* containing all pillars. By excluding from H_* this additional pillar and its two incident edges in H, we obtain a Hamiltonian path in G. □

From the case $r = 2$ in Lemma 2.1, we can analogously derive the following result.

Corollary 2.5. *The conclusions of Theorem 2.4 remain valid if $r = 2$, provided no quadrangle of the prism contains both pillars in question.*

We observe that the minimum-cardinality counterexamples to Tait's conjecture due to Holton and McKay [5, Figure 1.1] are special cases more generally captured by Corollary 2.5. We note in passing that in the case of $r = 2$ where the two pillars in question are part of a quadrangle of the prism, the inflated graph has an associated unique Hamiltonian cycle in $C_n \square K_2$; see Remark 1.

2.1 Counting Hamiltonian cycles after TF-inflation

Fix a prism $C_n \square K_2$ with even $n \geq 4$, and focus on its two meandering Hamiltonian cycles H_1 and H_2. Let F be a TF used in a TF-inflation of $C_n \square K_2$. There are two Hamiltonian paths in F starting at a and ending at c. Likewise, there are four Hamiltonian paths in F starting at b and ending at c. Correspondingly, we speak of an $[a, c]$-traversal, respectively of a $[b, c]$-traversal, of F. Now, if in the TF-inflation of $C_n \square K_2$ the Hamiltonian cycle H_1 is being expanded by an $[a, c]$-traversal of F, then the Hamiltonian cycle H_2 is being expanded by a $[b, c]$-traversal of F. Suppose G derives from $C_n \square K_2$ by q TF-inflations. Then, we have the following: If H_1 expands in q_1 copies of F by an $[a, c]$-traversal, then it expands in the remaining $q - q_1$ copies of F by a $[b, c]$-traversal. Correspondingly, H_2 expands in q_1 copies of F by a $[b, c]$-traversal, and it expands in the remaining $q - q_1$ copies of F by an $[a, c]$-traversal. It follows that H_1 yields $2^{q_1} \cdot 4^{q-q_1}$ Hamiltonian cycles in G, and H_2 yields $4^{q_1} \cdot 2^{q-q_1}$ Hamiltonian cycles in G. This gives a total number of

$$h(q, q_1) = 2^{q_1} \cdot 4^{q-q_1} + 4^{q_1} \cdot 2^{q-q_1}$$

Hamiltonian cycles. Using the arithmetic-geometric mean-inequality (respectively, a trivial upper bound) we obtain

$$2^{3q/2+1} = 2\sqrt{2^q 4^q} \leq h(q, q_1) \leq 2^{2q+1}.$$

2.2 Recovering Tutte's counterexample

A closer look at Theorem 2.4 shows that it examines—up to a subsequently described contraction—generalizations of Tutte's counterexample to n-prisms for arbitrary odd n. Let us apply Theorem 2.4 to the prism $C_3 \square K_2$ specifically, subject to the following conditions.

(i) The three TF-inflations are performed at the three vertices of the base cycle (s_0, s_1, s_2).
(ii) The resulting graph G^* should be as symmetrical as possible; this means that, without loss of generality, G^* should contain in particular the edges $s_0^b s_1^a$, $s_1^b s_2^a$, and $s_2^b s_0^a$.

In fact, the symmetry group of G^* is the cyclic group of order three, and it is fixed-point-free. However, the top cycle is still a triangle Δ. By contracting Δ, the resulting graph or its mirror image is the usual presentation of Tutte's counterexample to Tait's conjecture.

3 Conclusion

By combining the n-sided prism with the operation of TF-inflation, we constructed an infinite family of Hamiltonian planar 3-connected cubic graphs (Theorem 2.4 and Corollary 2.5, case n even), respectively such non-Hamiltonian graphs (corresponding odd case). This construction was carried out under the condition that the 2-valent vertex c must be incident with the corresponding pillar. Consequently, Tutte's counterexample to Tait's conjecture arises as a minimal element (in the case $r > 2$) after contracting the top triangle, whereas Holton and McKay's counterexample to Tait's conjecture appears as a minimal element in the case $r = 2$; see Corollary 2.5. In other words, our results demonstrate that these counterexamples are not isolated coincidences; rather, they are in a certain sense just minimal elements in the corresponding partial order of counterexamples. On top of this, we also showed how divergent the number of Hamiltonian cycles may be depending on the applied TF-inflations.

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