# Towards the Solution of Robust Gas Network Optimization Problems Using the Constrained Active Signature Method

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## ABSTRACT

This work studies robust gas network optimization under uncertainties in demand and in the physical parameters. The corresponding optimization problems are nonconvex in node pressures and flows along the pipes. They are thus very difficult to solve for realistic instance sizes. In recent approaches, an adaptive bundle method has been developed, where one solves the occurring adversarial problems via iteratively refined piecewise linear relaxations. These subproblems need to be solved always from scratch using mixed-integer linear programming (MIP). As alternative to the MIP solver, we employ here a nonsmooth optimization approach that allows a warm start strategy such that it can profit from the results obtained for coarser relaxations. We evaluate the approach for realistic gas network topologies and outline possibilities for future research.

## 1 INTRODUCTION

Resource-efficient distribution of energy is one of the grand challenges of modern times. In the current transformation of the energy system, natural gas is considered as a transition technology that is used to ensure stable and resilient energy supply. Furthermore, in the future current energy sources may be combined or even replaced by hydrogen. Optimization of the respective energy networks is of major importance [5].

In addition, optimization of gas network operation should be hedged against uncertainties that are inherent in the energy demands and in the physical parameters of gas transport. In particular when demand distributions are unknown (e. g. if they are market-driven) or when uncertainties cannot be measured easily (which for example is true for the roughness in the pipes), protection is sought in the sense of robust optimization. A particular mathematical challenge consists in the fact that the relation between gas pressure at the network nodes and gas flow along the pipes is nonconvex quadratic. Thus, in order to determine a best possible operation of the active elements in the network such as compressors and valves, a robust nonconvex optimization problem needs to be solved. To this end, several robust optimization approaches have been established, see, e.g., [1, 12]. An efficient solution approach is given by an adaptive bundle method [7] that can cope with the nonconvexities. It is integrated within an outer-approximation scheme that is able to decide on discrete as well as continuous decisions for operating the active elements [8]. To solve the adversarial problems that arise within the bundle method, the nonconvex expressions are replaced by iteratively refined piecewise linear relaxations. The latter are modeled via a mixed-integer linear optimization problem (MIP). In each iteration, the MIP is refined until a predefined error guarantee on the quality of the relaxation is given. Typically, after applying small changes in a MIP, the corresponding simplex based branched-and-bound approaches do not allow any warm start strategy as they usually cannot profit from earlier iterations. Therefore, in [7], the MIPs are always solved from scratch using available MIP solvers.

In this work, we advance this method by replacing the MIP solver by an approach called CASM for Constrained Active Signature Method. CASM is tailored to solve optimization problems where the objective function as well as the constraints are continuous and piecewise linear. In contrast to the MIP solvers that always need to solve the problems from scratch, the CASM approach allows a warm start based on the optimization results obtained for a coarser approximation of the nonconvex expressions. The goal of this work is to apply CASM to the problem of robust operation of gas network operation in order to evaluate its applicability.

The structure of this paper is as follows. In the next section, the considered problem stemming from gas transport is introduced. Section 3 presents CASM in more detail including also a description of the warm start option. Numerical results are discussed in Sec. 4. Section 5 contains a conclusion and an outlook.

## 2 THE GAS TRANSPORT PROBLEM

We consider a problem that arises in the context of gas networks, namely the stationary robust gas transport problem. For a profound explanation of models and solution approaches for gas transport problems, we refer to [5].

Here, we consider the problem of finding an optimal control that is robustly protected against the perturbation of physical parameters. We aim for a minimum-cost control of compressors. Constraints are thereby that all demands should be satisfied and no physical constraints should be violated. The control of active elements can be modeled as here-and-now variables at the first stage and the realization of physical states as wait-and-see variables at the second stage. The realization of physical states takes place after uncertain parameters realize themselves. As uncertain parameters, we consider demands and pressure loss coefficients, where the latter is due to uncertain frictions of the pipes. For every possible realization of the pressure loss coefficients, physical feasibility of the gas transport has to be maintained by the network operator. In the following, we model the arising robust gas transport problem from the point of view of the network operator.

We describe a gas network by a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{A})$, where the arcs model pipes and compressors ($\mathcal{A} = \mathcal{A}_{pi} \cup \mathcal{A}_c$) and an incidence matrix $A \in \{-1, 0, 1\}^{|\mathcal{V}|\times|\mathcal{A}|}$. We denote the gas flow by $q \in \mathbb{R}^{|\mathcal{A}|}$, where its sign indicates the flow's direction. Further, squared pressure values are denoted by $\pi \in \mathbb{R}^{|\mathcal{V}|}$. To ensure uniqueness of the physical states, we fix the pressure value at one so-called root node. By $w(\Delta)$, we denote the costs of a control $\Delta$ of compressors. We use a linear compressor model, so that a value $\Delta_a$ induces a pressure increase of $\Delta_a$ at compressor $a$. In total, we consider the following robust optimization problem:

$$\begin{aligned}
\min_{\Delta\in[\underline{\Delta},\overline{\Delta}]} \max_{(d,\lambda)\in\mathcal{U},\pi,q} \quad & w(\Delta) + \sum_{v\in V} \max\{0, \underline{\pi}_v - \pi_v, \pi_v - \overline{\pi}_v\} && \text{(1a)}\\
\text{s.t.} \quad & Aq = d && \text{(1b)}\\
& (A^T\pi)_a = \Delta_a \qquad \forall a \in \mathcal{A}_c && \text{(1c)}\\
& (A^T\pi)_a = -\lambda_a q_a |q_a| \qquad \forall a \in \mathcal{A}_{pi} && \text{(1d)}\\
& (q, \pi) \in \mathbb{R}^{|\mathcal{A}|} \times \mathbb{R}^{|\mathcal{V}|}. && \text{(1e)}
\end{aligned}$$

Thereby, the uncertainty set $\mathcal{U}$ is defined as follows:

$$\mathcal{U} := \{(d, \lambda) \mid \lambda \in [\underline{\lambda}, \overline{\lambda}], d_i \in [\underline{d}, \overline{d}], \sum_{i=1}^{n} d_i = 0\}.$$

Fixing the uncertain parameters to some values, there is a unique physical state, i.e., unique flow and pressure variables, that fulfills the physical constraints [1, 2]. Due to this fact, we can reformulate (1) as a box-constrained optimization problem writing the pressure as a function of the other parameters (see [7]):

$$\min_{\Delta\in[\underline{\Delta},\overline{\Delta}]} \max_{(d,\lambda)\in\mathcal{U}} \quad w(\Delta) + \sum_{v\in V} \max\{0, \underline{\pi}_v - \pi_v(\Delta; d, \lambda), \pi_v(\Delta; d, \lambda) - \overline{\pi}_v\}.$$

In [7], an adaptive bundle method is developed to solve problems of this kind. For this purpose, the bundle method is applied to the outer minimization problem with the optimal value function of the inner maximization problem as objective function. As in every iteration of the bundle method, an approximate function evaluation is required, the inner maximization problem has to be approximately solved in every iteration. This inner adversarial problem is the following nonconvexly constrained optimization problem:

$$\begin{aligned}
\max_{(d,\lambda)\in\mathcal{U},\pi,q} \quad & \sum_{v\in\mathcal{V}} \max\{0, \underline{\pi}_v - \pi_v, \pi_v - \overline{\pi}_v\} && \text{(2a)}\\
\text{s.t.} \quad & Aq = d && \text{(2b)}\\
& (A^T\pi)_a = \Delta_a \qquad \forall a \in \mathcal{A}_c && \text{(2c)}\\
& (A^T\pi)_a = -\lambda_a q_a |q_a| \qquad \forall a \in \mathcal{A}_{pi} && \text{(2d)}\\
& (q, \pi) \in \mathbb{R}^{|\mathcal{A}|} \times \mathbb{R}^{|\mathcal{V}|}. && \text{(2e)}
\end{aligned}$$

In [7] this adversarial problem is approximately solved via piecewise linear relaxation. The adaptive bundle method only allows for a certain error in the optimal objective value. As a relaxation that fulfills a requested error bound, for each of the pressure loss constraints, piecewise linear relaxation via the delta method [1, 3, 9] is used. In the adaptive bundle method [7], an error bound on the optimal objective value of the adversarial problem is requested and a consequent bound for the error in the pressure loss constraints is provided. As this theoretical bound turned out to be not very tight, the strategy in [7] is to allow for large errors in the constraints and to refine in case of a too large a posteriori error in the objective (cf. [7, Section 5.1.1, 5.1.2]). In [7], it is noted that the run time of the bundle method is largely determined by the solution of the adversarial problem up to the requested error, where the piecewise linearly relaxed adversarial problems are solved via MIP solvers. This motivates the development and analysis of an alternative solution strategy for these piecewise linear problems in the present paper. In particular, as by the use of the refinement strategy, sequences of refined relaxations are solved, a method that allows for warm start strategies has the potential to speed up computations.

## 3 THE OPTIMIZATION APPROACH CASM

In [6] the so-called Constrained Active Signature Method (CASM) for solving constrained piecewise linear optimization problems was introduced and analyzed in detail. Therefore, here we just introduce its main aspects briefly. Based on results contained, e.g., in [11], it follows that any continuous piecewise linear function $f : \mathbb{R}^n \mapsto \mathbb{R}$, $y = f(x)$, can be represented by a system of equations of the form

$$\begin{aligned}
z &= c + Zx + Mz + L|z| \,,\\
y &= d + a^\top x + b^\top z \,,
\end{aligned}$$

where $z \in \mathbb{R}^s$ is the vector of so-called switching variables, $c \in \mathbb{R}^s$, $Z \in \mathbb{R}^{s\times n}$, strictly lower triangular matrices $M, L \in \mathbb{R}^{s\times s}$ , $d \in \mathbb{R}$, $a \in \mathbb{R}^n$, $b \in \mathbb{R}^s$. Here and throughout, $|z|$ denotes the componentwise absolute value of the vector $z$. Using a similar representation also for piecewise linear constraints and ignoring a possible constant shift in the objective, the piecewise linearly relaxed adversarial problems to be maximized for the function evaluations in the bundle method can be described by

$$\begin{aligned}
\max_{x\in\mathbb{R}^n, z\in\mathbb{R}^s} \quad & a^\top x + b^\top z\\
\text{s.t.} \quad & 0 = g + Ax + Bz + C|z|\\
& 0 \geq h + Dx + Ez + F|z|\\
& z = c + Zx + Mz + L|z| \,,
\end{aligned} \qquad (3)$$

with additional constants $g \in \mathbb{R}^m, h \in \mathbb{R}^p, A \in \mathbb{R}^{m\times n}, B, C \in \mathbb{R}^{m\times s}, D \in \mathbb{R}^{p\times n}$ and $E, F \in \mathbb{R}^{p\times s}$ to describe the $m$ piecewise linear equality and $p$ piecewise linear inequality constraints. For each $x$, we define the signature vector

$$\sigma(x) = (\mathrm{sign}(z_i(x)))_{i=1\dots s} \in \{-1, 0, 1\}^s \ .$$

The signature vectors yield the inverse images

$$P_\sigma \ \equiv \ \{x \in \mathbb{R}^n : \mathrm{sign}(z(x)) = \sigma\} \quad \text{for} \quad \sigma \in \{-1, 0, 1\}^s \ ,$$

which are relatively open polyhedra that form collectively a disjoint decomposition of $\mathbb{R}^n$. The signatures $\sigma \in \{-1, 1\}^s$ are called definite and the associated $P_\sigma$ are by continuity open.

Any uniformly convex continuous objective function must attain a unique minimizer $x_\sigma$ on each one of the closed sets $\bar{P}_\sigma$. Furthermore, for a given definite $\sigma$, the optimization problem (3) restricted to the corresponding $\bar{P}_\sigma$ is smooth. These observations motivated the optimization strategy of CASM. To obtain a strictly convex continuous objective function, a quadratic regularization term is added to the target function in Eq. (3). Subsequently, standard KKT theory for the resulting smooth constrained quadratic optimization problem on $\bar{P}_\sigma$ can be applied yielding a system of $n + 2s + m + p$ linear equations and $n + 2s + m + p$ unknowns as necessary optimality conditions. It can then be verified by checking the signs of the corresponding Lagrange multipliers if the solution of this system of equations is indeed a minimizer of the original problem. If this is not the case, the computed solution can be used to determine a descent direction and also a new polyhedron $P_{\tilde{\sigma}}$ to be considered. When changing from $P_\sigma$ to $P_{\tilde{\sigma}}$ one component $\sigma_i$ of $\sigma$ changes its sign such that at a point $x \in \bar{P}_\sigma \cap \bar{P}_{\tilde{\sigma}}$ one must have for the signature vector that $\sigma_i(x) = 0$ and $z_i(x) = 0$. Such a switching variable $z_i(x)$ is called active and the corresponding absolute value evaluation yields a nonsmooth contribution.

Hence, the nonsmooth optimization algorithm CASM solves a sequence of quadratic optimization problems where in each iteration of CASM, a linear system $Mv = w$ has to be solved where $M$ is a usually very sparse $(n + 2s + m + p) \times (n + 2s + m + p)$ matrix with real-valued entries. Due to the guaranteed existence of a minimizer there always exists a solution $v$ of the linear system but it must not be unique. Note, that $v$ is related to a Newton step. All remaining steps in the algorithm are rather cheap linear algebra operations.

The convergence properties of the nonsmooth optimizer CASM are analyzed in [6] including also the derivation of optimality conditions for piecewise linear constrained optimization problems. These results extend the work on the unconstrained case presented in [4]. Since the convergence analysis in both papers is based on KKT theory, both algorithms terminate at local optimizers. In the nonconvex case it is not ensured that a global solution is found.

CASM expects a feasible starting point and feasibility is maintained throughout. In the example from the gas market considered here, a feasible starting point can be constructed in various ways. We consider cascades of up to three successive relaxed adversarial problems that result from the application of the bundle method to problem (1). Hence, we need a feasible starting point for each of the three different relaxations. We have no special previous knowledge for the starting point of an optimization cascade, i.e., the coarsest relaxation. Therefore, we determine a starting point by using the nominal values for $d$ and $\lambda$. If these two variables are fixed, the physical states, i.e., the pressure $\pi$ and the flow $q$, are uniquely determined as described in Section 2. Hence, they can be evaluated.

For the finer discretizations, i.e., the next two optimization tasks in the provided cascade, a warm start strategy will be used, which is an essential aspect of this paper. In the bundle method (cf. Sec. 2), the MIP is solved anew after each refinement, without the old solution having any influence since so far no warm start strategy is known for MIP solvers. In contrast to that, we perform a warm start when using CASM for the inner loop. That is, if the inner problem is solved for a given discretization, a new starting point for the next model with a finer discretization is calculated with the help of the previous solution. For this purpose, the calculated values for demand $d$ and pressure loss coefficient $\lambda$ are taken from the solution and new starting values for pressure and flow are determined for the refined model. This step coincides with the one for the starting point, i.e., the coarsest discretization.

## 4 NUMERICAL RESULTS

In this section, we present numerical results for the application of CASM to the adversarial problem of the robust gas transport problem. We hence determine the worst-case values of uncertain parameters for a given compressor control.

*GasLib-Instances.* We use data from a library of realistic gas network instances [10]. In detail, we conduct numerical experiments on the instances GasLib-11, GasLib-40 and GasLib-134, which model gas networks with 11, 40 and 134 nodes, respectively. For the network with 40 nodes, we distinguish between a non robust feasible and a robust feasible control. GasLib-134 thereby models the Greek gas network. For all test cases, we consider in the end a piecewise linear relaxation of the adversarial problem (2).

We investigate different choices of the compressor control $\Delta$. First, we use arbitrarily chosen controls, which are not robust feasible. Second, we consider a robust feasible control implying that the optimal value of the adversarial problem is equal to 0. Furthermore, we use different choices of the piecewise linear relaxation. That is, we impose different allowed errors up to which the relaxed problem deviates from the original one in terms of the nonconvex pressure loss constraints leading to different discretizations in the piecewise linear approximation of the nonconvex term. As described in Sec. 2, the error bounds are possibly refined during one iteration of the applied bundle method. Therefore, we investigate here the applicability of CASM for such a cascade of refinements.

In detail, we solve the adversarial problem for GasLib-11 with an initial compressor control for two typical sizes of given error bounds. The adversarial problems for GasLib-40 are taken from runs of the adaptive bundle method. First, we applied the adaptive bundle method with an uncertainty set for demand $d$ and pressure loss coefficients $\lambda$ that is $[0.95d, 1.05d] \times [\lambda, 1.1\lambda]$. For this case, multiple refinements of the relaxation of the adversarial problem are requested in the bundle method's last iteration, when a robust feasible compressor control is investigated. To this series of adversarial problems, we applied CASM. Second, we enlarged the uncertainty set to $[0.9d, 1.1d] \times [\lambda, 1.5\lambda]$. In this case, multiple refinements are requested in an earlier iteration of the bundle method in which the compressor control is not robust feasible. We used these data for another series of adversarial problems to which we applied CASM. The adversarial problems for GasLib-134 are also taken from a run of the adaptive bundle method, namely for the

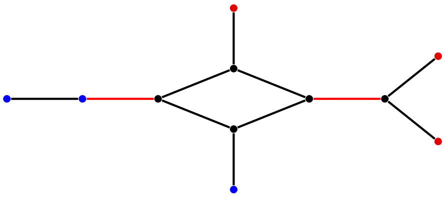

Figure 1: Topology of GasLib-11

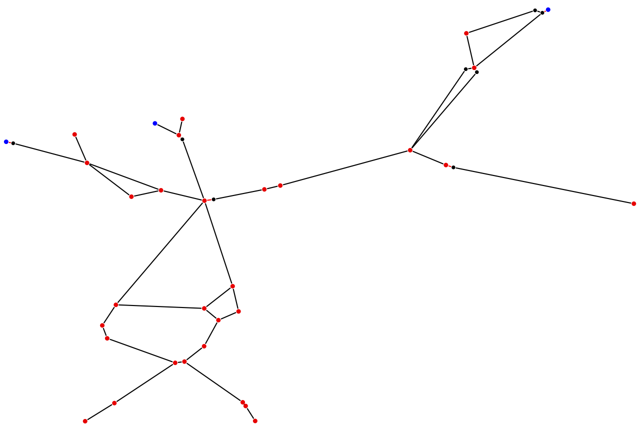

Figure 2: Topology of GasLib-40

| | GasLib-11 | | GasLib-40 non robust feasible | | | GasLib-40 robust feasible | | |
|---|---|---|---|---|---|---|---|---|
| relaxation | 1. | 2. | 1. | 2. | 3. | 1. | 2. | 3. |
| variables $n$ | 44 | | 170 | | | | | |
| equal. const. $m$ | 19 | | 54 | | | | | |
| inequal. const. $p$ | 70 | | 314 | | | | | |
| switching variables $s$ | 175 | 183 | 331 | 341 | 573 | 315 | 315 | 319 |
| rows/columns of eq. system | 484 | 500 | 1206 | 1226 | 1690 | 1174 | 1174 | 1182 |
| iterations | 65 | 23 | 939 | 556 | 731 | 472 | 193 | 204 |

**Table 1: Complexity of different GasLib instances for their respective optimizations and iterations needed by CASM.**

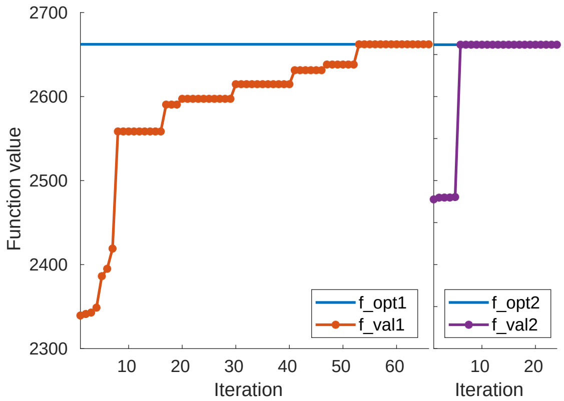


**Figure 3: Optimization history of CASM for the GasLib-11 instance. Left: Coarse discretization. Right: Fine discretization.**

uncertainty set $[0.8d, 1.2d] \times [\lambda, 2\lambda]$. Again, we applied CASM to multiple refinements that are requested for a compressor control that is not robust feasible. Sketches of the topologies of GasLib-11 and GasLib-40 can be seen in Figs. 1 and 2, respectively. There, sources where gas can be injected are depicted in blue, sinks where gas can be extracted in red, inner nodes, pipes and short pipes in black and compressor stations in red.

*Complexity & results.* As described in Sec. 3, for CASM the numbers $n$ of variables, $m$ of equality constraints, $p$ of inequality constraints and $s$ of absolute value evaluations are of primary importance for the complexity of the optimization problem. As explained briefly in Sec. 3 the size of the system of equations that must be solved in each iteration depends linearly on each of these numbers. For the different GasLib instances and their respective refinements, the corresponding numbers are given in Tab. 1, where the first column states the GasLib instance and the third one the relaxations. Note that the number of constraints is linear in the number of edges and nodes of the underlying model. As can be seen from this table, the improved approximations of the nonconvex term, i.e., the finer discretizations, only influence the number $s$ of switching variables. The growth in the size of the system matrix fits perfectly to the number $(n+2s+m+p)\times(n+2s+m+p)$ stated in Sec. 3. The numbers of iterations needed by CASM are stated in the last column.

For the GasLib-11 instance, Fig. 3 shows the development of the function values during the optimization runs using CASM for the two discretizations of the nonconvex function. The red line with the label `f_val1` depicts the function values for the coarse discretization and the purple line (label `f_val2`) the function values for the fine discretization. The blue line labeled with `f_opt` illustrates the globally optimal function value, which is reached in both cases and changes from the first optimization to the second one by less than 0.5. The values of the respective optimal variables vary also. During the last iterations, there is a very small increase in the quadratic regularization term that is added to the piecewise linear objective in the CASM, see Sec. 3. However, the value of the nonregularized objective function remains constant, a fact that could be used for an improved termination criterion in the future.

The optimal values of the variables obtained from the optimization for the coarse discretization do not provide a feasible starting point for the fine optimization. Therefore, it is necessary to determine a new feasible starting point for CASM. A corresponding approach exploiting the results from the optimization for the coarse discretization was described at the end of Sec. 3. Figure 3 shows that this warm start strategy yields a larger initial function value for the finer discretization.

For the GasLib-40 instance, we proceed in a similar way. As before, we consider one of the adversarial problems from the bundle method, where subsequently refinements are made yielding a cascade of two refinements, i.e., three optimization runs are performed. From Tab. 1 we see, especially for the instance GasLib-40 with a non robust feasible compressor control, that the refinements induce

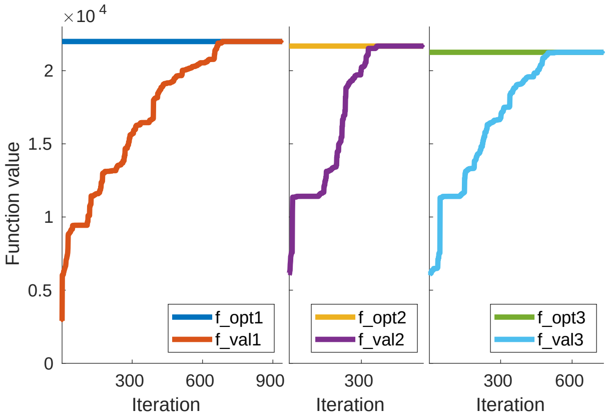


**Figure 4: Optimization history of CASM for the non robust feasible GasLib-40 instance**

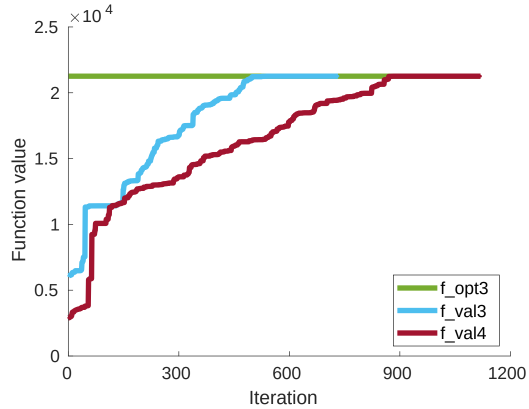


**Figure 5: Comparison of the third optimization for non robust feasible GasLib-40 with and without warm start**

a significant increase in the dimension of the system of equations and lead also to a noticeable effect on the run times. Despite the fact that the models become more complex with each refinement, the number of iterations does not increase in the same way.

Figure 4 illustrates the development of function values for the three optimization runs in red, purple and blue, respectively (cf. `f_val`). The constant lines depict the globally optimal function values that can be achieved for the respective discretization (cf. `f_opt`). Once more, the objective value at the starting point for the next level of discretization is larger than the initial value for the previous one.

To emphasize the effect of the warm start strategy, a comparison with an optimization not using the warm start is shown in Fig. 5 for the finest discretization from the non robust feasible GasLib-40 instance. The blue graph (`f_val3`) shows again the development of the function values using the warm start option. The dark red graph (`f_val4`) shows the optimization history of the function values for the case that an initial point is determined without exploiting the previously performed optimization such that the function value at the initial point is much smaller. In this case, 1118 iterations are necessary and hence less than for the three individual optimizations in total. However, the size of the system of equations is larger such that one iteration is much more expensive. In addition, from a technical point of view, the considered model with the adapted finer discretization can only be generated if the solution of the previous one is known. Otherwise, a refinement that leads to the same a posteriori error would be even more complex to solve (cf. Sec. 2) supporting also the warm start strategy proposed here.

Next, we consider the GasLib-40 instance with a robust feasible compressor control yielding for the first two relaxations of the three-part cascade the optimal function value 0.6965 and for the finest one the optimal function value 0 corresponding to a robust feasible compressor control. Here, CASM needs 472 iterations to solve the first model with the coarsest discretization to reach a local optimum that is not globally optimal. However, since CASM determines only locally optimal points this fits to the theoretical analysis of CASM as described in Sec. 3. The same behavior is observed also for the second relaxation, where the number of iterations is clearly reduced

| | GasLib-134 | | |
|---|---|---|---|
| relaxation | 1. | 2. | 3. |
| variables $n$ | 534 | | |
| equal. const. $m$ | 230 | | |
| inequal. const. $p$ | 784 | | |
| switching variables $s$ | 737 | 1107 | 1985 |
| rows/columns of eq. system | 3022 | 3762 | 5518 |
| iterations | 869 | 327 | 328 |
| solves | 902 | 327 | 328 |

**Table 2: Overview of the complexity of the GasLib-134 instances for their respective optimizations and number of iterations and solves needed by CASM.**

by the warm start (cf. Tab. 1). In the third optimization, however, the global optimum is found again.

As last test case, we consider the much larger GasLib-134 instance. Tab. 2 states the problem size and the numbers of iteration needed by CASM. The last line gives the number of equation solves required. In contrast to the previous instances, in some iterations the system of equations was not solved accurately enough resulting in an increase of the parameter in front of the quadratic regularization term to improve the conditioning of the system matrix $M$. This causes the difference between the number of iterations and the number of equation solves.

For this instance, the development of the function values is shown in Fig. 6. It is particularly noticeable that after the warm start in the second and third optimization, again the iterates just increase the value of the quadratic penalty term and keep the value of the nonregularized objective function constant. That is we observe the same behavior as for the GasLib-11 instance. Solving the finest discretization without a warm start, analogous to the GasLib-40 instance, the function value would be 361 at the initial iterate and 6339 iterations and 7727 equation solves are needed to reach the global optimum.

Finally, we discuss the sparsity structure of the matrix $M$ of the linear system $Mv = w$ in more detail. Fig. 7 shows the nonzero entries of the matrix $M$ in the first iteration of the first optimization of the GasLib-11 instance, where `nz` indicates the total number of nonzero entries of $M$. Almost all rows have between two and four nonzero entries. However, there are also some rows with up to 44 nonzero entries. These rows are directly related to the relaxation of Eq. (2d) since for the relaxation more evaluation of the absolute

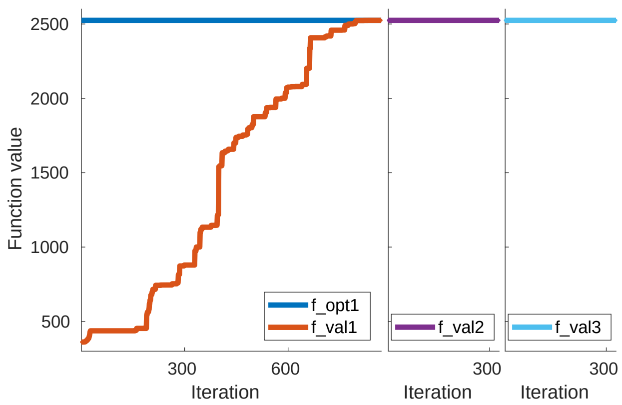


**Figure 6: Optimization history of CASM for the non robust feasible GasLib-134 instance**

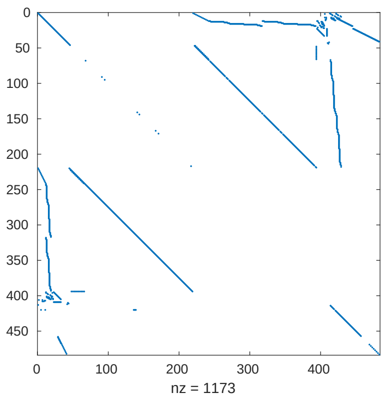


**Figure 7: Sparsity of the matrix $M$ in the first iteration of the first optimization for the GasLib-11 instance**

value function are needed. Hence, $M$ is denser in these areas. This sparsity of the matrix $M$ is currently not exploited in the implementation. Therefore, we do not compare computation times with other solvers here but just state that the state-of-the-art MIP solver Gurobi, which is implemented in C, is in the range of seconds when solving the optimization problems, whereas a Matlab implementation of CASM is in the range of seconds for the GasLib-11. For the GasLib-40 instances, the current implementation of CASM needed a few minutes. For the optimizations of the GasLib-134 instance the solution was obtained after a bit less than 2 hours. Since solving the system of equations currently accounts for about 95% of the computing time exploiting sparsity could reduce these run times considerably.

## 5 CONCLUSION AND OUTLOOK

In this work, we have shown that adversarial problems in robust gas transport optimization can be solved with a warm start strategy obtained from nonsmooth optimization. This is a major advantage, as typically MIP solvers cannot profit from earlier iterations, if only a small part of the model is changed. The warm start ability is very advantageous if the piecewise linear relaxation of a nonconvex adversarial problem is required to have high quality, i.e., when many iterative refinements are necessary in order to approximate well the nonconvex functions via piecewise linear functions. Whereas currently the corresponding series of mixed-integer linear problems is always solved from scratch, CASM promises the ability of a warm start strategy such that a subsequent iteration profits from earlier ones, without the necessity to always start from scratch. This will be very helpful for large robust gas network instances.

However, it is necessary to improve the run times needed by CASM via algorithmical engineering in order to allow meaningful comparisons with other solvers. An essential aspect is to speed up the solution of the system of equations that is required in each iteration of CASM. The system matrix is usually very sparse such that sparse solvers would reduce the run time consierably. As can be seen from the numerical results, CASM reached the global optimum in three out of four test cases. Further studies are required to analyse this behaviour in more detail and also the impact on the outer optimization performed by the bundle method. Especially in the scenario considered here, where one solves a cascade of optimization problems with moderate refinements when going from one level to the next one, it should be possible to develop a globalisation strategy to ensure that a global optimum is reached which is required by the adaptive bundle method. Another goal will be to integrate CASM directly into the bundle method. This will also allow a more rigorous comparison to the MIP based approach.

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## REFERENCES

[1] D. Aßmann, F. Liers, and M. Stingl. 2019. Decomposable robust two-stage optimization: an application to gas network operations under uncertainty. *Networks* 74, 1 (2019), 40–61. https://doi.org/10.1002/net.21871

[2] M. Collins, L. Cooper, R. Helgason, J. Kennington, and L. LeBlanc. 1977/78. Solving the pipe network analysis problem using optimization techniques. *Management Sci.* 24, 7 (1977/78), 747–760. https://doi.org/10.1287/mnsc.24.7.747

[3] B. Geißler, A. Martin, A. Morsi, and L. Schewe. 2012. Using piecewise linear functions for solving MINLPs. In *Mixed integer nonlinear programming*. IMA Vol. Math. Appl., Vol. 154. Springer, New York, 287–314.

[4] A. Griewank and A. Walther. 2019. Finite convergence of an active signature method to local minima of piecewise linear functions. *Optim. Methods Softw.* 34, 5 (2019), 1035–1055. https://doi.org/10.1080/10556788.2018.1546856

[5] T. Koch, B. Hiller, M.E. Pfetsch, and L. Schewe (Eds.). 2015. *Evaluating gas network capacities*. MOS-SIAM Series on Optimization, Vol. 21. SIAM.

[6] T. Kreimeier, A. Walther, and A. Griewank. 2021. An active signature method for constrained abs-linear minimization. (2021). https://opus4.kobv.de/opus4-trr154/frontdoor/index/index/docId/474

[7] M. Kuchlbauer, F. Liers, and M. Stingl. 2021. Adaptive bundle methods for nonlinear robust optimization. *Informs Journal on Computing* (2021). https://opus4.kobv.de/opus4-trr154/frontdoor/index/index/docId/307

[8] M. Kuchlbauer, F. Liers, and M. Stingl. 2021. Outer approximation for mixed-integer nonlinear robust optimization. (2021). https://opus4.kobv.de/opus4-trr154/frontdoor/index/index/docId/414

[9] H.M. Markowitz and A.S. Manne. 1957. On the solution of discrete programming problems. *Econometrica* 25 (1957), 84–110. https://doi.org/10.2307/1907744

[10] M.E. Pfetsch, A. Fügenschuh, B. Geißler, N. Geißler, R. Gollmer, B. Hiller, J. Humpola, T. Koch, T. Lehmann, A. Martin, A. Morsi, J. Rövekamp, L. Schewe, M. Schmidt, R. Schultz, R. Schwarz, J. Schweiger, C. Stangl, M.C. Steinbach, S. Vigerske, and B.M. Willert. 2015. Validation of nominations in gas network optimization: models, methods, and solutions. *Optim. Methods Softw.* 30, 1 (2015), 15–53. https://doi.org/10.1080/10556788.2014.888426

[11] S. Scholtes. 2012. *Introduction to piecewise differentiable equations*. Springer, New York. https://doi.org/10.1007/978-1-4614-4340-7

[12] J. Schweiger and F. Liers. 2018. A decomposition approach for optimal gas network extension with a finite set of demand scenarios. *Optim. Eng.* 19, 2 (2018), 297–326. https://doi.org/10.1007/s11081-017-9371-4