Conic Linear programming games

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ABSTRACT

This paper introduces a new class of cooperative games whose characteristic function is the optimal value of a conic program. There are two main reasons motivating this new class of games. On the one hand, this class allows to model an appealing model of cooperation in portfolio selection but in our way to this analysis prove that this class is a natural extension of the important class of linear production games (see Owen (1982)[8]). In addition, we show several applications by choosing the appropriate cones and in particular using second order cone constraints we present new cost sharing results on the classical portfolio selection problem and on location, covering, packing and TSP cooperative games.

KEYWORDS

cooperative games, linear production games, conic programming

1 INTRODUCTION

The interest in analyzing cost sharing in different models of mathematical programming models has motivated an increasing number of papers modeling different aspects of cooperation. The interaction between operations research and game theory is a fruitful research field nowadays. Classical operational research models deal with problems in which a decision maker aims to design a program for optimizing the operation of a complex system. On the other side, game theory is concerned with decision problems in which several decision makers interact. Clearly, in order to model and solve complex systems in which several decision makes interact, both the models and methodologies of operations research and game theory are needed. In this context many classes of operations research games arise like, for instance, inventory games (see Meca et al. (2003)[7]), network games (see Bergantiños et al. (2014)[1], Perea et al. (2009)[9], Puerto et al. [10, 11]), and queueing games (see Timmer and Scheinhardt (2013)[12]). Borm et al. (2001) [2] provides a survey of this field.

This paper elaborates on this direction. Our goal is to introduce a class of cooperative games whose characteristic function is the optimal value of a conic program. There are two main reasons motivating this new class of games. On the one hand, this class allows to model an appealing model of cooperation in portfolio selection but in our way to this analysis we have realized that this class is a natural extension of the important class of linear production games (see Owen (1982)[8]). In a linear production game a set of agents pool their available resources to produce some goods through a production procedure that can be modeled as a linear programming problem. Using a duality argument, Owen proved that each linear production game is totally balanced and found a family of core allocations for this class of games. Linear production games have been widely studied. An introduction to linear production games can be found in González-Díaz (2010)[6].

Our initial motivation was to consider a family of mathematical programming games in which players deal with linear production constraints with an extra set of quadratic restrictions. This type of problems fits within a more general class of mathematical programming games where the characteristic function of a coalition is the optimal value of a conic linear program. Players cooperate by pooling their resources on the linear conic constraints. We prove that under some standard conditions this class of games is totally balanced. Then, we show several applications by choosing the appropriate cones and in particular using second order cone constraints we present new cost sharing results on the classical portfolio selection problem.

2 THE CLASS OF CONIC LINEAR PROGRAMMING COOPERATIVE GAMES

Let $N = \{1, \dots, n\}$ be a set of agents. Each of them has a bundle of m commodities $b(i) = (b_k(i))_{k \in \{1,\dots,m\}}$ and we denote by $b(S) = \sum_{i \in S} b(i)$. Each agent $i \in N$ is interested in optimizing the following program:

$$\begin{aligned} \max\ & \langle P, X \rangle \\ \text{s.t. } & \langle A_k, X \rangle = b_k(i),\ k = 1, \dots, m \\ & X \in K. \end{aligned} \tag{Conic}$$

As it is usual in conic programming K is a pointed, closed, convex cone with non-empty interior and X is the set of variables of the problem and let K^* be the dual cone of K. Classic examples of the above model are the standard linear programming where $K = \mathbb{R}^n_+$, second order cone programming where $K = \{x \in \mathbb{R}^n : x_n \geq \sqrt{\sum_{j=1}^{n-1} x_j^2}\}$, semidefinite programming where K is the cone of symmetric positive semidefinite matrices of order n or $K = K_1 \oplus K_2 \oplus \ldots \oplus K_p$ being K_s, $s = 1, \dots, p$ pointed, closed, convex cones with non-empty interior. Conic programming is a relatively novel area of mathematical programming that can be solved efficiently and it is already implemented in modern solvers.

One can now consider the conic dual of (Conic) which is given by:

$$\begin{aligned} \min\ & \sum_{k=1}^{m} y_k b_k(i) \\ \text{s.t. } & \sum_{k=1}^{m} y_k A_k - H = P, \\ & H \in K^*. \end{aligned} \tag{Conic Dual}$$

It is well-known that if one of $\mathcal{F} := \{X \in K : \langle A_k, X \rangle = b_k(i),\ k = 1, \dots, m\}$ or $\mathcal{F}^* = \{(y, H) : \sum_{k=1}^{m} y_k A_k - H = P, H \in K^*\}$ has non-empty interior then the optimal values of both problem coincide (zero duality gap).

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If the agent of a set $S \subseteq N$ cooperate then they can pool their resources being $b(S) = \sum_{i \in S} b(i)$ and the problem and its dual to be considered by the coalition are

$$\begin{aligned} v(S) = \max\ & \langle P, X\rangle && \text{(Conic-S)} \\ \text{s.t. } & \langle A_k, X\rangle = b_k(S),\ k = 1, \ldots, m \\ & X \in K. \end{aligned}$$

$$\begin{aligned} d(S) = \min\ & \sum_{k=1}^{m} y_k b_k(S) && \text{(Dual-S)} \\ \text{s.t. } & \sum_{k=1}^{m} y_k A_k - H = P, \\ & H \in K^*. \end{aligned}$$

The natural questions arise. Under which conditions the agents in N are willing to cooperate and how to allocate the surplus of this cooperation among them.

Next, we define the game (N, v) with set of players N and characteristic function $v(S)$ given as the optimal value of problem (Conic-S). One can prove that the game (N, v) is balanced so that cooperation in this situation is naturally enforced. Recall that the core of a game (N, v) consists of those allocations which divide the benefit of the grand coalition, $v(N)$, in such a way that any other coalition receives at least its value by the characteristic function. Formally,

$$Core(N, v) = \left\{x \in \mathbb{R}^n \,/ x(N) = v(N) \text{ and } x(S) \geq v(S) \text{ for all } S \subset N\right\}.$$

Theorem 2.1. *Let $H(N) \in \overset{\circ}{K^*}$ and $y(N)$ be an optimal solution of the dual problem for the grand coalition N. The allocation $x_i = \sum_{k=1}^{m} y_k(N) b_k(i)$, for all $i = 1, \ldots, n$ belongs to the core of (N, v).*

3 THE PORTFOLIO SELECTION GAME

One of the most interesting problems in finance and modern Operations Research is the classical Markowitz portfolio model and its many variants. In this situation an agent i owns a capital amount c_i that is willing to invest on a number of assets $j = 1, \ldots, m$ with an expected return μ_j and covariance matrix $Q \in \mathbb{R}^{m \times m}$ positive definite. The goal of the investors (agents) is to determine the amount x_j to invest in asset j for all $j = 1, \ldots, m$ in order to maximize his expected return while limiting the risk by a threshold $\sigma^2(i)$. Therefore, agent i looks to optimize the following problem.

$$\begin{aligned} \max\ & \sum_{j=1}^{m} \mu_j x_j && \text{(Portfolio)} \\ \text{s.t. } & \sum_{j=1}^{m} x_j = c_i, \\ & x^t Q x \leq \sigma^2(i), \\ & x \geq 0. \end{aligned}$$

Although at a first glance the reader may think that this problem does not fit the previous framework it is not the case.

Indeed, the covariance matrix Q admits a Cholewsky factorization $Q = R^t R$ where R is triangular matrix. Thus, $x^t Q x = y^t y$ with $y = Rx$ and problem (Portfolio) becomes

$$\begin{aligned} \max\ & \sum_{j=1}^{m} \mu_j x_j \\ \text{s.t. } & \sum_{j=1}^{m} x_j = c_i, \\ & \sum_{j=1}^{m} r_{kj} x_j - y_k = 0,\ k = 1, \ldots, m \\ & s_1 \leq \sigma^2(i) \\ & \sum_{j=1}^{m} y_j^2 \leq s_1 \\ & s_1 \geq 0, x \geq 0, y \text{ free.} \end{aligned}$$

Finally, this problem can be written as:

$$\max \sum_{j=1}^{m} \mu_j x_j \qquad \text{(SOC-Portfolio)}$$

$$\text{s.t. } \sum_{j=1}^{m} x_j = c_i, \qquad (1)$$

$$\sum_{j=1}^{m} r_{kj} x_j - y_k = 0,\ k = 1, \ldots, m \qquad (2)$$

$$s_1 \leq \sigma^2(i) \qquad (3)$$

$$(y, s_1) \in SOC_{m+1} := \{(y, s) \in \mathbb{R}^{m \times 1} : \sqrt{\sum_{j=1}^{m} y_j^2} \leq s_1\},\ x \in \mathbb{R}^m_+.$$

Introducing dual variables u_0 and u_{m+1} for constraints (1) and (2); and u_k $k = 1, \ldots, m$ for constraints (3) the conic dual of this problem reads as:

$$\begin{aligned} \min\ & u_0 c_i + u_{m+1} \sigma^2(i) \\ \text{s.t. } & u_0 + \sum_{k=1}^{m} u_k r_{kj} \geq \mu_j,\ j = 1, \ldots, m \\ & -u_k + h_k = 0,\ k = 1, \ldots, m, \\ & u_{m+1} - h_{m+1} = 0, \\ & (h_1, \ldots, h_m, h_{m+1}) \in SOC_{m+1} \\ & u_0, u_k,\ k = 1, \ldots, m \text{ free }, u_{m+1} \geq 0. \end{aligned}$$

The above problem is equivalent to the following:

$$\begin{aligned} \min\ & u_0 c_i + u_{m+1} \sigma^2(i) \\ \text{s.t. } & u_0 + \sum_{k=1}^{m} u_k r_{kj} \geq \mu_j,\ j = 1, \ldots, m \\ & \sum_{j=1}^{m} u_j^2 \leq u_{m+1} \\ & u_k,\ k = 0, \ldots, m \text{ free.} \end{aligned}$$

We observe that if a set of agents S decides to cooperate they put together their cash $c(S) = \sum_{i \in S} c_i$ and assume a level of risk given by the function $\sigma^2(S)$ and the corresponding primal and dual problems for the coalition S are:

$$\max \sum_{j=1}^{m} \mu_j x_j \qquad \text{(SOC-Portfolio-S)}$$

$$\text{s.t. } \sum_{j=1}^{m} x_j = c(S), \qquad (4)$$

$$\sum_{j=1}^{m} r_{kj}x_j - y_k = 0,\ 1 \le k \le m \quad (5)$$

$$s_1 \le \sigma^2(S) \quad (6)$$

$$(y, s_1) \in SOC_{m+1},\ x \in \mathbb{R}^m_+.$$

$$\begin{aligned} \min\ & u_0 c_i + u_{m+1}\sigma^2(S) \\ \text{s.t. } & u_0 + \sum_{k=1}^{m} u_k r_{kj} \ge \mu_j,\ j = 1, \ldots, m \\ & \sum_{j=1}^{m} u_j^2 \le u_{m+1} \\ & (u_0, u_1, \ldots, , u_m, u_{m+1}) \in SOC_{m+1} \end{aligned}$$

The reader may note that the feasible region $\mathcal{F}^*(S)$ of the dual problems for all the coalitions $S \subseteq N$ are the same.

In the following, we will assume some condition on the function $\sigma^2(\cdot)$. Actually, it is natural to suppose that the larger the coalition the higher the risk that it is willing to accept. A natural condition of this kind is the following:

$$\frac{\sigma^2(R)}{|R|} \le \frac{\sigma^2(S)}{|S|},\ \forall R, S \subseteq N,\ \text{such that } |R| \le |S|. \quad (7)$$

Proposition 3.1. *Superadditivity under the condition (7).*

Theorem 3.2. *Assume that the dual problems for the grand coalition satisfies to be strictly feasible and the risk function σ^2 satisfies condition (7), then cooperative portfolio selection game is balanced.*

4 THE BINARY AND CONTINUOUS NON-CONVEX QUADRATIC MINIMIZATION GAME

Now, we consider allocation problems built on one the most challenging problems in mathematical programming, namely the binary and continuous non-convex quadratic minimization problem. Let us assume that we are given general symmetric $m \times m$ matrix Q a $\ell \times m$ matrix A and ℓ-vectors b_r for all $r \in N$. As usual, we denote by $b(R) = \sum_{r \in R} b_r$.

We consider the cooperative game (N, c) with characteristic function $c(R)$ for any coalition $R \subseteq N$. defined as the optimal value of the following problem:

$$\begin{aligned} c(R) = \min\ & x^t Q x + 1/2 c^t x \\ \text{s.t. } & A x^t = b(R) \\ & x \ge 0,\ x_j \in \{0, 1\},\ \forall j \in B \end{aligned}$$

Recall that $A \in mathbbR^{n \times n}$ is a completely positive matrix if it can be factorized as $A = B^t B$ where B (non-necessarily squared) has all its entries non-negative. and $C^*_{1+\ell}$ denotes the cone of completely positive matrices of order $1 + \ell$. We introduce the completely positive relaxation $\tilde{c}(R)$ of $c(R)$ which always provides a lower bound on its optimal value (see [3] for further details on this relaxation and its construction). For this purpose, we write the constraints $a_i^t X a_i = b_i^2(R)$ in its equivalent matrix form $\langle B_i, X \rangle = b_i^2(R)$ and $Diag(e_i)$ is a $\ell \times \ell$ diagonal matrix with 1 in the i-th position of the diagonal and zero everywhere else. Then, the relaxation is:

$$\begin{aligned} \tilde{c}(R) = \min\ & 1/2 c^t x + \langle Q, X \rangle \\ \text{s.t. } & A x^t = b(R) \\ & \langle B_i X \rangle = b_i^2(R), \quad i = 1, \ldots, \ell \\ & x_i - Diag(e_i) X = 0, \quad i = 1, \ldots, \ell \\ & \begin{pmatrix} 1 & x^t \\ x & X \end{pmatrix} \in C^*_{1+\ell}. \end{aligned}$$

The following result proves that under mild assumptions $\tilde{c}(R)$ is exact and provides the optimal value of $c(R)$.

Theorem 4.1. *Assume that the linear portion of $c(R)$, $L(R) = \{x \ge 0 : Ax = b(R)\}$ satisfies $x \in L(R) \Rightarrow 0 \le x \le 1,\ \forall j \in B$. Then, $\tilde{c}(R)$ is equivalent to $c(R)$:*

1. *$c(R) = \tilde{c}(R)$,*
2. *If (x^*, X^*) is an optimal solution for $\tilde{c}(R)$ then x^* is in the convex hull of optimal solutions of $c(S)$.*

The reader should observe that, as pointed out in [3], the condition in Therorem 4.1 is not restrictive since if $B \ne \emptyset$ and it is not already implied, then one can achieve it without loss of generality augmenting one constraint $x_j + s_j = 1$ where $s_j \ge 0$ is a slack variable for all $j \in B$.

Now, we consider the conic dual $\tilde{d}(R)$ of $\tilde{c}(R)$. For this construction, as it is usual, one has to associate variables u_i, $i = 1, \ldots, \ell$, to the first block of constraints, variables v_i, $i = 1, \ldots, \ell$, to the second block of constraints, variables w_i, $i = 1, \ldots, \ell$ to third block. Then, the dual problem becomes:

$$\begin{aligned} \tilde{d}(R) = \max\ & \sum_{i=1}^{\ell} u_i b_i(R) + \sum_{i=1}^{\ell} v_i b_i^2(R) \\ \text{s.t. } & u^t A + \sum_{i=1}^{\ell} w_i \le 1/2 c^t \\ & \sum_{i=1}^{\ell} v_i B_i - \sum_{i=1}^{\ell} w_i Diag(e_i) + S = Q \\ & u, v, w \text{ free }, S \in \mathcal{CP}_{\ell}. \end{aligned}$$

Under the condition that the feasible region of the dual $\tilde{d}(R)$ has non-empty interior the dual has optimal solution and $\tilde{c}(R) = \tilde{d}(R)$. Since the feasible region of the dual problems for the different coalitions $\tilde{d}(R')$ is always the same, the same conclusion also applies to all of them and thus, $\tilde{c}(N) = \tilde{d}(N)$ as well.

Following the same construction as in the previous section, let us denote by u^*, v^*, w^* and S^* an optimal solution of $\tilde{d}(N)$. We define the following allocation of the cost of the grand coalition $c(N)$:

$$x_r = (u^*)^t b_r + (v^*)^t b_r^2, \quad \forall r \in N. \quad (8)$$

It is clear, by the form of the objective function of $\tilde{d}(N)$ that under the hypothesis that we are assuming

$$\begin{aligned} c(N) &= \sum_{r \in N} x_r \\ &= \sum_{i=1}^{\ell} u_i^* (\sum_{r \in N} b_{ri}) + \sum_{i=1}^{\ell} v_i^* (\sum_{r \in N} b_{ri}^2) \end{aligned}$$

Then, we are in position to state the result that proves the stability of allocation (8).

Theorem 4.2. *The allocation $(x_r)_{r \in N}$ defined in (8) satisfies $\sum_{r \in R} x_r \le c(R)$ for all $R \subset N$.*

This result has interesting structural implications since it informs on the structure of core allocations based on dual prices of conic programs. Nevertheless, although it is constructive it requires solving conic problem on the cone of copositive matrices. The cones of copositive and completely positive matrices do allow self-concordant barrier functions, but these functions can not be evaluated in polynomial time. Thus, the classical interior point methodology does not work. Optimizing over these cones is thus NP-hard, and restating a problem as an optimization problem over one of these cones does not resolve the difficulty of that problem ([4]). Thus, resulting in NP-hard problems. In any case, even though it may be difficult solving $\tilde{d}(N)$ for finding the dual multipliers we have proven that dual based core allocations are also possible beyond the class of linear production games and its relatives. It also explains similar results already known for some other classes of games as: location packing and covering or TSP.

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