Challenges in System Reliability and its application in Network Optimization

Guillermo Rela
Facultad de Ingeniería. Universidad de la República
Montevideo, Uruguay
grela@fing.edu.uy

Franco Robledo
Facultad de Ingeniería. Universidad de la República
Montevideo, Uruguay
frobledo@fing.edu.uy

Pablo Romero
Facultad de Ingeniería. Universidad de la República
Montevideo, Uruguay
promero@fing.edu.uy

ABSTRACT

Computer Science deals with search, optimization and decision problems. System Reliability does not fall into the previous categories; however, it is strictly related to them. We are given a multi-component system subject to random failures on its components, and the goal is to find a number, called the *reliability* of the system, which represents the probability of correct operation. As a consequence, Reliability Analysis is a branch of knowledge with a strong interplay with Optimization.

In this paper, we state major challenges in the analysis and design of stochastic binary systems (SBS). We review the Network Utility Problem (NUP), where the goal is to maximize the network utility subject to a constraint in the size of the network. Our contributions are two-fold: first, we fully characterize optimal network topologies using an alternative framework of SBS. Then, we introduce a level of separability, which provides hints for the design of network with maximum utility.

KEYWORDS

Network Utility Problem, Network Optimization, SBS.

1 INTRODUCTION

In system reliability analysis, the goal is to find the probability of correct operation of the system subject to random failures. In the most celebrated models, such as the *all-terminal reliability* model, finding the reliability both efficiently and exactly is an utopia unless $\mathcal{P} = \mathcal{NP}$, since Arnie Rosenthal mathematically proved its $\mathcal{NP}$-Hardness [15]. As a consequence, the research community focused on the design of pointwise reliability estimations and/or exact reliability evaluation for special topologies [7].

The maximum reliability evaluation among all (p, q)-graphs (i.e., graphs with p nodes and q links) is a challenging problem with partial progress and several open problems [4]. Topological network optimization subject to a minimum reliability constraint with dependent failures is a recent problem under study [2]. The Network Utility Problem (NUP) is a combinatorial optimization problem inspired by reliability. Curiously enough, networks with maximum utility accept both an efficient and exact reliability evaluation, such as trees and elementary cycles. The interplay between the NUP and system reliability has its origin in the preliminary work [6]; however, it is not largely explored.

In this paper, a novel interplay between the NUP and system reliability is introduced. The contributions can be summarized in the following items:

- We fully characterize special systems, called *separable systems* under the all-terminal and source-terminal communication model.
- We discuss a novel interplay between separable systems and the NUP, with potential applications to network design.

The document is organized in the following manner. Section 2 formally presents the Network Utility Problem, and its interplay with the all-terminal reliability model. Section 3 summarizes the main concepts in system reliability, in terms of Stochastic Binary Systems (SBS). We fully characterize special SBS, called *separable systems* under the source-terminal model, in Section 4. Section 5 makes an analysis of separable systems, while Section 6 focus in separability in graphs. In Section 6.1, the interplay between the NUP and separable systems is discussed. Section 7 briefly indicates concluding remarks and trends for future work.

2 NETWORK UTILITY PROBLEM

In topological network design, the network connectivity is a key aspect [16]. If n entities should be connected by certain links to be selected, the link connectivity, $\lambda(G)$, is the minimum number of links that should be removed from G in order to disconnect G. To increase $\lambda(G)$, we must deploy and pay for new links. Consider a network represented by a graph G with m links and n nodes:

- The *profit* of the network G is its link-connectivity, $\lambda(G)$, and
- The *cost* is the number of links exceeding the lower bound $n-1$, precisely, $m-n+1$.

The co-rank of a graph G is precisely $c(G) = m - n + 1$. The *network utility* is the profit-cost difference: $u(G) = \lambda(G) - c(G)$. The Network Utility Problem is defined in [6] as follows:

$$\max_{G \subseteq K_n} u(G) \tag{1}$$

$$s.t. \tag{2}$$

$$\lambda(G) \geq k \tag{3}$$

The cheapest network design meeting $\lambda(G) = k$ must be a k-regular graph, whenever $k > 1$. Frank Harary provided a construction of a family of k-regular graphs with n nodes, $H_{n,k}$, now called *Harary graphs* [11].

There are at least two open problems related with the NUP. Even though Harary graphs provide the optimal solution, we do not know a full list of optimal solutions for any fixed value of k and n. Furthermore, among the set of graphs with maximum utility for any fixed k and n, it is interesting to find which one has the maximum reliability. In the following paragraphs we define the all-terminal reliability model.

Assume we are given a connected graph G whose n nodes are perfect but its m links may fail independently with identical probability $q = 1 - p$. We aim to find the probability that the random graph is connected, called *all-terminal reliability* and denoted by $R_G(p)$. Let us denote by F_i the number of connected subgraphs with precisely $m - i$ links. They all share the same probability $p^{m-i}(1-p)^i$. Summing the probability of all the disjoint events, we obtain

$$R_G(p) = \sum_{i=0}^{m} F_i p^{m-i}(1-p)^i. \quad (4)$$

Therefore, finding $R_G(p)$ is equivalent to counting subgraphs to obtain all entries of the F-vector, $F = (F_0, \ldots, F_m)$. This counting problem is called network reliability analysis problem, and it remains at the heart of network reliability theory [1]. Some work performed by Michael Ball and Scott Provan established that these counting tasks belong to the $\#\mathcal{P}$-Complete class [13]. By its definition, it is worth noticing that:

- $F_i = \binom{m}{i}$ for all $i < \lambda(G)$;
- $F_i = 0$, for all $i > m - n + 1$.

Therefore, the number of unknowns is $(m - n + 1) - \lambda + 1 = -u(G) + 1$. This curious interplay shows that the graphs with maximum utility require the minimum reliability computation.

3 STOCHASTIC BINARY SYSTEMS

The following terminology is adapted from [1].

Definition 3.1 (Stochastic Binary System). A *stochastic binary system* is a triad (S, r, ϕ):

- $S = \{1, \ldots, N\}$ is a ground set of *components*,
- $r = (r_1, \ldots, r_N)$ are their *elementary reliabilities*, and
- $\phi : \{0,1\}^N \to \{0,1\}$ is the *structure function*.

The concept of reliability is generalized to arbitrary stochastic binary systems.

Definition 3.2 (Reliability/Unreliability). Let $\mathcal{S} = (S, r, \phi)$ be a stochastic binary system, and consider a random vector $X = (X_1, \ldots, X_N)$ with independent coordinates governed by Bernoulli random variables such that $P(X_i = 1) = r_i$. The *reliability* of $\mathcal{S}$ is the probability of correct operation of the system:

$$R_{\mathcal{S}} = P(\phi(X) = 1) = E(\phi(X)) = \sum_{x:\phi(x)=1} P(X = x). \quad (5)$$

The *unreliability* of $\mathcal{S}$ is $U_{\mathcal{S}} = 1 - R_{\mathcal{S}}$.

A stochastic binary system is *homogeneous* if the elementary reliabilities are identical (i.e., $r_i = r$ for all i). In this paper we deal with homogeneous SBSs.

Definition 3.3 (Pathsets/Cutsets). Let $\mathcal{S} = (S, r, \phi)$ be a stochastic binary system. A possible state or configuration $x \in \{0,1\}^N$ is a *pathset* (resp. *cutset*) if $\phi(x) = 1$ (resp., if $\phi(x) = 0$).

The binary set $\{0, 1\}$ is equipped with the partial order, defined by $0 \leq 0$, $0 \leq 1$ and $1 \leq 1$. The set $\{0,1\}^N$ inherits a natural order in the Cartesian product. Given two partially ordered sets A and B, a function $f : A \to B$ is monotonically increasing if $f(a_1) \leq f(a_2)$ whenever $a_1 \leq a_2$. As usual, we denote $y < x$ if $y \leq x$ and $y \neq x$. Let us denote by $\vec{0}_N$ (resp. $\vec{1}_N$) the binary word with all bits set to 0 (resp. to 1), and by δ_i the binary word with all bits in 0 except the bit in position i which is set to 1.

Definition 3.4 (Stochastic Monotone Binary System (SMBS)). The triad $\mathcal{S} = (S, r, \phi)$ is a *stochastic monotone binary system* if the structure function $\phi : \{0,1\}^N \to \{0,1\}$ is monotonically increasing, $\phi(\vec{0}_N) = 0$ and $\phi(\vec{1}_N) = 1$.

Definition 3.5 (Minpaths/Mincuts/Rays). Let $\mathcal{S} = (S, r, \phi)$ be an SMBS:

- A pathset x is a *minpath* if $\phi(y) = 0$ for all $y < x$.
- A cutset y is a *mincut* if $\phi(x) = 1$ for all $x > y$.
- The x-ray is the set $S_x = \{y \in \{0,1\}^N : y \geq x\}$.

An SMBS is fully characterized by its mincuts (or its minpaths). In fact, if we are given the complete list of minpaths, then the complete list of pathsets is precisely the union of the x-rays over all minpaths x.

We will denote by $\overline{x}$ the state complementary to x in bits (i.e., 0 in x are set to 1 in $\overline{x}$, and vice-versa). In particular, $\overline{\phi(x)} = 1 - \phi(x)$. The following definition of duality will be useful for our later analysis of monotonicity and bounds [14]:

There exists a direct connection between SBSs and propositional logic. Recall that a theorem-proving procedure is the first $\mathcal{NP}$-Complete decision problem established by Stephen Cook [9]. In other words, the recognition of a tautology is a hard decision problem from propositional logic.

Theorem 3.6. *The reliability evaluation of an arbitrary SMBS belongs to the class of $\mathcal{NP}$-Hard problems.*

Proof. Arnie Rosenthal proved that the reliability evaluation for the K-terminal reliability model belongs to the class of $\mathcal{NP}$-Hard computational problems [15]. Since K-Terminal is a particular SMBS, the result follows by inclusion. □

Corollary 3.7. *The reliability evaluation of an arbitrary SBS belongs to the class of $\mathcal{NP}$-Hard problems.*

4 SEPARABLE SYSTEMS

Observe that $\{0,1\}^N$ is the set of the extremal points of the unit hypercube $Q_N \subseteq \mathbb{R}^N$. Let us assign labels to the extremal points of Q_N according to a given structure ϕ. Every hyperplane defines a partition of $\mathbb{R}^N$ into two subsets. Consider the family of hyperplanes $\mathcal{H}$ such that $\vec{0}_N$ and $\vec{1}_N$ lie on different sides. For any member H of $\mathcal{H}$, denote by $Q_0 \subseteq Q_N$ the extremal points of the hypercube that belong to the side of $\vec{0}_N$; and $Q_1 = Q_N - Q_0$. Define a structure function ϕ_H such that its cutsets are precisely Q_0, and its pathsets are Q_1. Consider an equivalence relation $(\mathcal{H}, \sim)$ such that $H_1 \sim H_2$ if and only if $\phi_{H_1} = \phi_{H_2}$.

Recall that in the Euclidean space $\mathbb{R}^N$, a hyperplane is fully characterized by a normal vector $\vec{n}$ and a point P that belongs to the hyperplane: $\langle \vec{n}, X - P \rangle = 0$, where $\langle x, y \rangle = \sum_{i=1}^{N} x_i y_i$ is the inner product. If we denote $\vec{n} = (n_1, \ldots, n_N)$ and $\langle \vec{n}, P \rangle = \alpha_0$, the hyperplane can be written as $\sum_{i=1}^{N} n_i x_i = \alpha_0$. By convention and without loss of generality, we will consider that cutsets lie on the hyperplane or in its negative side, so that they verify $\sum_{i=1}^{N} n_i x_i \leq \alpha_0$, and that pathsets lie on the positive side of the hyperplane, and verify $\sum_{i=1}^{N} n_i x_i > \alpha_0$.

Lemma 4.1. *Consider a monotone structure ϕ. If $\phi = \phi_H$ for some hyperplane H, then there exists $H' \sim H$ with non-negative normal vector such that $\|\vec{n}\|_1 = \sum_{i=1}^{N} n_i = 1$.*

Proof. Let $\phi = \phi_H$ for the hyperplane $H) \sum_{i=1}^{N} n_i x_i = \alpha_0$, and suppose that there exists some index j such that $n_j < 0$. There are two exhaustive and disjoint cases:

i There exists some mincut $x = (x_1, \ldots, x_N)$ such that $x_j = 0$: in this case, we know that $x + \delta_j$ is a minpath, so, $\phi(x +$

$\delta_j) = 1$. By the definition of the separating hyperplane and the structure function ϕ_H, we get that $\sum_{i=1}^N n_i x_i \leq \alpha_0$ but $\sum_{i=1}^N n_i x_i + n_j > \alpha_0$. The only possibility is that $n_j > 0$. But we assumed $n_j < 0$; this is a contradiction.

ii All mincuts verify $x_j = 1$: Consider an alternative hyperplane $H' : \sum_{i \neq j}^N n_i x_i = \alpha_0 - n_j$. We will prove that $H' \sim H$. If x is a mincut, then $\sum_{i=1}^N n_i x_i \leq \alpha_0$, and therefore $\sum_{i \neq j}^N n_i x_i \leq \alpha_0 - n_j$. If x is a minpath, then $x_j = 1$. Since $\sum_{i=1}^N n_i x_i > \alpha_0$ we get that $\sum_{i \neq j}^N n_i x_i > \alpha_0 - n_j$. Observe that $n_j = 0$ in the new hyperplane H', and $H' \sim H$ as desired.

By an iterative replacement of all the negative coordinates we obtain an equivalent hyperplane $H' \sim H$ with non-negative vector $\vec{n'}$, expressed by $H' : \sum_{i=1}^N n'_i x_i = \alpha'$ for some real number α'. Finally, observe that $\vec{0}_N$ is always a cutset, so $0 \leq \alpha'$. Analogously, $\vec{1}_N$ is always a pathset, so $\sum_{i=1}^N n'_i > \alpha' \geq 0$. The result is obtained by a normalization of the resulting vector, which is possible since $\sum_{i=1}^N n'_i > 0$. □

Even though there exist infinite equivalent hyperplanes, using Support Vector Machine (SVM) it is possible to find a single hyperplane with the largest gap (this is, with the largest distance to any of the vertices in the hypercube). Using Lemma 4.1, we can replace it by an equivalent hyperplane with non-negative vector. Without loss of generality, we will assume a non-negative normal vector with unit 1-norm.

Proposition 4.2. *The structures ϕ_H are monotone.*

Proof. By Lemma 4.1, in particular we can choose $n_i \geq 0$ in the hyperplane $H : \sum_{i=1}^N n_i x_i = \alpha_0$. Let us denote $f(x) = \sum_{i=1}^N n_i x_i$. If $x \leq y$, then $f(x) \leq f(y)$, and therefore $\phi_H(x) \leq \phi_H(y)$. □

A subtlety is that the *mincuts* from Lemma 4.1 are indeed the points $Q_0 \subset Q_N$ that are closer to the original hyperplane. It is possible to prove that not all SMBSs can be represented by an hyperplane.

Definition 4.3 (Separable System). An SBS is *separable* if the cutsets/pathsets can be separated by some hyperplane.

An interpretation of separable systems recalls Riesz representation theorem for Hilbert spaces [8]. Indeed, the structure of a separable system can be written as an indicator that an inner-product exceeds some threshold in a Hilbert space:

$$\phi(x) = 1_{\langle x, \vec{n} \rangle \geq \alpha_0}. \quad (6)$$

A separable system can be represented with $N + 1$ real numbers, instead of a logic table of 2^N values for an arbitrary SBS. This compact, space-efficient representation is a key point of our interest in separable systems.

5 ANALYSIS OF SEPARABLE SYSTEMS

In this section, we first study the hardness of the reliability evaluation for separable systems. Then, we provide two alternative characterizations of these systems.

5.1 Complexity

Even though separable systems accept an efficient representation, their reliability evaluation is computationally hard:

Theorem 5.1. *The reliability evaluation of separable systems belongs to the class of $\mathcal{NP}$-Hard problems.*

Proof. By reduction from *PARTITION*. Consider an instance of natural numbers $A = \{a_1, \ldots, a_N\}$, and let $s = \sum_{i=1}^N a_i$ be the sum over the elements of the list. Let us define $n_i = \frac{a_i}{s}$, $n_{min} = \min_{i=1,\ldots,N}\{n_i\}$, and consider the separable systems $\mathcal{S}_1$ and $\mathcal{S}_2$:

(1) The separable system $\mathcal{S}_1$ characterized by the hyperplane $\sum_{i=1}^N n_i x_i = \frac{1}{2} + \frac{n_{min}}{2}$;
(2) The separable system $\mathcal{S}_2$ characterized by the hyperplane $\sum_{i=1}^N n_i x_i = \frac{1}{2}$;

Observe that the difference of the reliability of both systems evaluated at $p = 1/2$ is:

$$\begin{aligned} & R_{\mathcal{S}_2}(1/2) - R_{\mathcal{S}_1}(1/2) \\ & = P(\sum_{i=1}^N n_i x_i \geq \frac{1}{2}) - P(\sum_{i=1}^N n_i x_i \geq \frac{1}{2} + \frac{n_{min}}{2}) \\ & = P(\sum_{i=1}^N n_i x_i = \frac{1}{2}) \\ & = \frac{\#\{(x_1, \ldots, x_N) \in \{0,1\}^N : \sum_{i=1}^N n_i x_i = \frac{1}{2}\}}{2^N}, \end{aligned}$$

and the last number is positive if and only if there exists a subset $B \subseteq \{1, \ldots, N\}$ such that $\sum_{i \in B} n_i = \frac{1}{2}$. In that case, if we multiply on both sides by s we get that $\sum_{i \in B} a_i = \frac{s}{2}$, and the answer to *PARTITION* for the list A is YES. Otherwise, the answer to *PARTITION* is NO. Therefore, the reliability evaluation of separable systems is at least as hard as *PARTITION*, and it belongs to the class of $\mathcal{NP}$-Hard problems. □

5.2 Characterizations

A natural question is to characterize separable systems in terms of pathsets and cutsets. Let us denote $CH(\mathcal{P})$ and $CH(C)$ the convex hull of the pathsets and cutsets respectively.

Theorem 5.2. *An SBS is separable iff $CH(\mathcal{P}) \cap CH(C) = \emptyset$.*

Proof. If the intersection is empty, Hahn-Banach separation theorem for convex sets asserts that there exists a hyperplane H that separates those convex sets [8]. As a consequence, $\phi = \phi_H$ for some hyperplane H.

For the converse, we know that the SBS is separable. Therefore, there exists some hyperplane $H : \sum_{i=1}^N n_i x_i = \alpha_0$ such that $\sum_{i=1}^N n_i x_i \leq \alpha_0$ for cutsets, and $\sum_{i=1}^N n_i x_i > \alpha_0$ for pathsets. Suppose for a moment that $CH(\mathcal{P}) \cap CH(C) \neq \emptyset$. There exists some element $z \in \mathbb{R}^N$ such that:

$$z = \sum_{j=1}^l \alpha_j x^j = \sum_{k=1}^s \beta_k y^k, \quad (7)$$

for some states $x^1, \ldots, x^l \in \mathcal{P}$, $y^1, \ldots, y^s \in C$, and non-negative numbers such that $\sum_{j=1}^l \alpha_j = \sum_{k=1}^s \beta_k = 1$. If we denote $x^j = (x_1^j, \ldots, x_N^j)$ we know that $\sum_{i=1}^N n_i x_i^j > \alpha_0$. Therefore, for $z = (z_1, \ldots, z_N)$ we get that:

$$\begin{aligned} \sum_{i=1}^N n_i z_i &= \sum_{i=1}^N n_i (\sum_{j=1}^l \alpha_j x_i^j) \\ &= \sum_{j=1}^l \alpha_j [\sum_{i=1}^N n_i x_i^j] \\ &> (\sum_{j=1}^l \alpha_j) \alpha_0 = \alpha_0. \end{aligned}$$

Analogously, using the fact that $z = \sum_{k=1}^{s} \beta_k y^k$ we get that $\sum_{i=1}^{N} n_i z_i \leq \alpha_0$, which is a contradiction. Therefore we must have $CH(\mathcal{P}) \cap CH(C) = \emptyset$, and the result holds. □

By Proposition 5.2 we have a full geometrical characterization of separable systems, which accept an efficient representation.

In the following, we consider an alternative characterization, in terms of weighted cutsets and pathsets. Consider an arbitrary assignment $n_1, \ldots, n_N$ of non-negative numbers to the respective components of the system. The condition $\sum_{i=1}^{N} n_i x_i \geq \alpha_0$ for all the pathsets is equivalent to finding the pathset x with minimum-cost, $c(x) = \sum_{i:x_i=1} n_i$, and testing if $c(x) \geq \alpha_0$. Analogously, the condition $\sum_{i=1}^{N} n_i y_i < \alpha_0$ for all the cutsets is equivalent to testing whether the cutset y with minimum cost, $c(y) = \sum_{i:y_i=0} n_i$, satisfies the test $S - c(y) < \alpha_0$, where $S = \sum_{i=1}^{N} n_i$ is the cost of the global system. Observe that, for convenience, the cost of a cutset is defined as the sum of the components under failure. In particular, we get the following characterization of separable systems:

Theorem 5.3. *An SBS is separable if and only if there exists an assignment of non-negative costs to the components $\{n_i\}_{i=1,\ldots,N}$ such that $S < c(y) + c(x)$, being $c(x)$ and $c(y)$ the pathset/cutset with minimum cost respectively.*

Proof. First, let us assume that we have a separable SBS with hyperplane $\sum_{i=1}^{N} n_i x_i = \alpha_0$. Using the previous reasoning, the assignment $\{n_i\}_{i=1,\ldots,N}$ verifies $c(x) \geq \alpha_0$ and $S - c(y) < \alpha_0$. Therefore, $S < c(y) + c(x)$.

For the converse, let us fix $\alpha_0 = c(x)$, the pathset with minimum cost. Clearly, the specific pathset x meets the condition $\sum_{i=1}^{N} n_i x_i \geq \alpha_0$; in fact the equality is met. By its definition, the inequality holds for the other pathsets. Finally, we use the fact that $S < c(y)+c(x)$ to verify that the cutset with minimum-cost, y, meets the inequality $\sum_{i=1}^{N} n_i y_i < \alpha_0$. The inequality for the other cutsets is straightforward since y is a cutset with minimum-cost. Therefore, the SBS is separable, concluding the proof. □

6 SEPARABILITY IN GRAPHS

Our characterization of separable systems has a straightforward reading in the celebrated all-terminal reliability model.

Definition 6.1 (Separable Graph). A graph $G = (V, E)$ is *separable* if there exists an assignment of non-negative real numbers $n_1, \ldots, n_m$ to its m links, and there exists a threshold α such that $c(E') \geq \alpha$ for all $E' \subseteq E$ such that $G' = (V, E')$ is connected, and $c(E') < \alpha$ otherwise.

Let G be a connected graph. Recall that Kruskal algorithm provides efficiently the cost of the minimum spanning tree, $MST(G)$. Furthermore, the cutset with minimum-cost, $m(G)$, is obtained using Ford-Fulkerson algorithm. Therefore, the following corollary of Theorem 5.3 holds for graphs:

Corollary 6.2. *A graph is separable iff there exists a feasible assignment $\{n_i\}_{i=1,\ldots,N}$ to the links such that $S < MST(G)+m(G)$, being $MST(G)$ the cost of the minimum spanning tree, $m(G)$ the mincut with minimum capacity, and $S = \sum_{i=1}^{N} n_i$ the sum of the link weights.*

With the following lemmas, we will present a hereditary property of separable graphs, stated in Theorem 6.5. Consider a simple connected graph $G = (V, E)$. We will consider two different link additions:

- We denote $G_{in} = G + e_{in}$ to the resulting graph after the addition of an internal link $e_{in} = \{u_1, u_2\}$, where $u_1, u_2 \in V$.
- We denote $G_{out} = G + e_{out}$ to the resulting graph after the addition of an external link $e_{out} = \{u_1, u_2\}$, where $u_1 \in V$ but $u_2 \notin V$.

Observe that $G + e_{in}$ and G share an identical node-set V, while the node-set for $G + e_{out}$ is $V \cup \{u_2\}$.

Lemma 6.3. *If G is nonseparable then G_{out} is nonseparable.*

Proof. Suppose for a moment that there exists a feasible assignment $\{n_i\}_{i=1,\ldots,N+1}$ for G_{out}. Then:

$$\begin{aligned}(\sum_{i=1}^{N} n_i) + n_{N+1} &< MST(G_{out}) + m(G_{out}) \\ &= MST(G) + n_{N+1} + \min\{m(G), n_{N+1}\} \\ &\leq MST(G) + n_{N+1} + m(G),\end{aligned}$$

and $\{n_i\}_{i=1,\ldots,N}$ would be a feasible assignment for G, which is a contradiction. Therefore, G_{out} is nonseparable. □

It is also possible to prove that:

Lemma 6.4. *If G is nonseparable then G_{in} is nonseparable.*

Observe that Lemma 6.4 informally states that graphs with more density are nonseparable. Using the counter-reciprocal of Lemmas 6.3 and 6.4, we can prove:

Theorem 6.5. *Separability is a hereditary property in graphs.*

And analogously we can prove that:

Lemma 6.6. *If G is separable, G_{out} is also separable.*

Corollary 6.7. *Cycles with arborescences are separable graphs*

Proof. We know that elementary cycles are separable. The result follows by the addition of one or several trees hanging to different nodes from the first cycle. Supported by Lemma 6.6, the separability is preserved by the addition of those links. □

Figure 1 depicts Monma graphs. Clyde Monma et. al. used these graphs to design minimum cost biconnected metric networks [12]. They attain the maximum reliability among all the graphs with p nodes and $q = p + 1$ links [5].

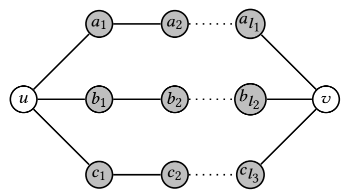

Figure 1: Monma graph M_{l_1+1,l_2+1,l_3+1}.

Lemma 6.8. *Monma graphs are nonseparable*

Proof. Consider an arbitrary order for the links of Monma graph, and the rule $\phi(x) = 1$ iff the Monma subgraph given by the links in x is connected. We will find a convex combination of pathsets and cutsets with identical result. Consider the four links $e_1 = \{u, a_1\}$, $e_2 = \{a_1, a_2\}$, $e_3 = \{u, b_1\}$ and $e_4 = \{b_1, b_2\}$ from Figure 1. Let us denote $1_{e_i,e_j}$ the binary word that is set to

1 in all the bits but 0 in the positions corresponding to the links e_i and e_j. Consider the following identity:

$$\frac{1}{2}(1_{e_1,e_2} + 1_{e_3,e_4}) = \frac{1}{4}(1_{e_1,e_3} + 1_{e_1,e_4} + 1_{e_2,e_3} + 1_{e_2,e_4}) \quad (8)$$

On one hand, we have a convex combination of cutsets. On the other, a convex combination of pathsets. By Theorem 5.2, Monma graphs are nonseparable. ◻

Recall that a node v in a graph G is a cut-point if $G - v$ has more components than G. A connected graph is biconnected if it has no cut-points. The addition of an ear in a graph G is the addition of an external elementary path between two different nodes from G. Frederickson-Jàjà characterization theorem asserts that there exists an ear decomposition of all biconnected graphs, such that $G = C_s \cup H_1 \cup H_2 \cup \cdots \cup H_r$, C_s is an elementary cycle and H_i is the addition of an ear to the previous graph [10]. This structural characterization of biconnected graphs lead us immediately to the following theorem that can be easily proved:

Theorem 6.9. *All biconnected graphs that are not elementary cycles are nonseparable.*

An analogous reasoning leads to the following generalization:

Corollary 6.10. *Two kissing cycles are nonseparable.*

A further generalization recalls Frederickson-Jàjà characterization: G is a bridgeless graph if and only if $G = C_s \cup H_1 \cup H_2 \cup \cdots \cup H_r$, C_s is an elementary cycle and H_i is the addition of an ear or a kissing cycle to the previous graph.

We are in conditions to fully characterize separable graphs:

Theorem 6.11. *A graph G is separable iff G falls into one of the four categories:*

(1) *G is not connected;*
(2) *G is a tree;*
(3) *G is an elementary cycle; or*
(4) *G is an elementary cycle with arborescences.*

Proof. In order to prove the converse, we test case by case that the graph is separable:

(1) If G is disconnected, all of its configurations are cutsets and the reliability is null. In this case, the inequality $\sum_{i=1}^{N} x_i > 2N$ is not satisfied by any binary vector $x = (x_1, \ldots, x_N)$, and the graph is separable.
(2) If G is a tree T_N with N links, the evidence is the hyperplane $\sum_{i=1}^{N} x_i \geq N$ (or a unit-assignment is feasible).
(3) If $G = C_N$ is an elementary cycle, the evidence is the inequality $\sum_{i=1}^{N} x_i \geq N - 1$.
(4) If G is a tree with arborescences, Lemma 6.7 states that G is separable.

◻

If we consider the case of a graph $G = (V, E)$ in which two nodes s, t called respectively *source* and *terminal* have to be connected, it is possible to prove that if G, a source terminal graph, is nonseparable then G_{in} neither, and G_{out} is also nonseparable.

Lemma 6.12. *Elementary cycles in a source terminal graph G are nonseparable.*

Proof. Suppose an elementary cycle and two nodes s and t source and terminal respectively, connected by two paths with n and r nodes respectively.

Consider an assignment of non-negative real numbers $\alpha_{1_1}, \ldots, \alpha_{1_n}$ to its n links, and $\alpha_{2_1}, \ldots, \alpha_{2_r}$ to its r links Let's call α_{1_m} and α_{2_m} to the smallest values in each path. Without loss of generality, let us assume that

$$\sum_{i=1}^{n} \alpha_{1_i} \leq \sum_{i=1}^{r} \alpha_{2_i}, \qquad \alpha_{1_m} \leq \alpha_{2_m}$$

Then, the separability condition is void if:

$$\sum_{i=1}^{n} \alpha_{1_i} + \sum_{i=1}^{r} \alpha_{2_i} \leq \sum_{i=1}^{n} \alpha_{1_i} + \alpha_{1_m} + \alpha_{2_m}$$

And so,

$$\sum_{i=1}^{r} \alpha_{2_i} \leq \alpha_{1_m} + \alpha_{2_m}$$

And this only can be true if α_{2_m} is the only link in this path. An analogous reasoning holds for the case where $\alpha_{2_m} \leq \alpha_{1_m}$. ◻

We are in conditions to fully characterize separable graphs:

Theorem 6.13. *A source terminal graph G is separable iff G falls into one of the two categories:*

(1) *there is no elementary path between s and t; or*
(2) *there is exactly one elementary path between s and t.*

Observe that the resulting graph G can have cycles, but they cannot include the distinguished nodes s and t.

Corollary 6.14. *The all-terminal reliability evaluation of separable graphs belong to the class $\mathcal{P}$ of polynomial-time problems.*

Proof. The analysis is straightforward. Let G be a separable graph:

(1) If G is not connected $R(G) = 0$.
(2) If $G = T_N$ a tree with N links with independent reliabilities $(p_e)_{e \in T_N}$ then $R(G) = \prod_{e \in T_N} p_e$.
(3) If $G = C_N$,
$$R(C_N) = \prod_{e \in C_N} p_e + \sum_{e \in C_N} (1 - p_e) \prod_{e' \neq e} p_{e'}.$$
(4) Finally, if G is an elementary cycle with arborescences: $G = C_l \cup T_s$, being T_s union of trees pending from the cycle C_l. Therefore, $R(G) = R(C_l) \times \prod_{e \in T_s} p_e$.

The reader can appreciate that the reliability computation is a product, or a sum of products of the elementary link reliabilities. Therefore, the number of operations involved are linear, or quadratic, in the number of links. ◻

The corank of a graph is the number of independent cycles. In a connected graph with n nodes and m links, its corank is precisely $c(G) = m - n + 1$. It is worth to remark that Theorem 6.13 can be re-stated in terms of corank: a connected graph G is separable if and only if its corank is either 0 or 1.

We close this section discussing an interplay between a combinatorial optimization called the Network Utility Problem (NUP) and separable graphs. First, observe that an arbitrary spanning tree of a connected graph G has $n - 1$ links. Therefore, the corank of a graph is precisely the number of *additional links that we must pay* to build the graph G, starting from a minimally-connected graph. In terms of communication, the corank of G represents *redundancy.* At the cost of redundancy, the resulting network can be robust under a certain amount of link failures. The profit is the link connectivity $\lambda(G)$, which represent the lowest number of links that should be removed in order to disconnect G. As a consequence, the *utility* of a graph, $u(G)$, is the difference between the connectivity and the corank: $u(G) = \lambda(G) - c(G) = \lambda - m + n - 1$. In [6], the authors formally proved the following

THEOREM 6.15. *The graphs with maximum utility are trees and cycles. Their utility value is* 1. *There is no other graph with maximum utility.*

COROLLARY 6.16. *All the graphs with maximum utility are separable graphs.*

The all-terminal reliability polynomial under identical elementary reliabilities in the links r is

$$R_G(r) = \sum_{i=\lambda(G)}^{c(G)-1} n_i(G)p^{m-i}(1-p)^i + \tau(G)p^{n-1}(1-p)^{m-n+1}, \quad (9)$$

being $n_i(G)$ the number of connected subgraphs of G with precisely $m - i$ links, and $\tau(G)$ the tree-number of G, which is known using Matrix-Tree Kirchhoff theorem [3]. Therefore, the number of unknowns is precisely the number of terms involved in the summation: $c(G) - \lambda$. The only cases where there are no terms in the sum occur either when $c(G) - \lambda = -1$, exactly in trees and cycles, or when $c(G) - \lambda = 0$, only in an elementary cycle with arborescence, K_4, Kite-graph and Bowtie-graph [6]. These graphs are considered as the simplest in terms of reliability analysis. Indeed, in [6] the authors define the *level of difficulty* of a graph as the difference $d(G) = c(G) - \lambda - 1$, and a graph is easy if and only if $d(G) \leq 0$:

COROLLARY 6.17. *All separable graphs are easy graphs.*

The reader can observe that the graphs with maximum utility $u(G)$ are the easiest graphs, with the minimum level of difficulty $d(G)$.

6.1 Discussion

A natural extension of our prior analysis is a classification of nonseparable systems.

Let $\mathcal{S} = (S, r, \phi)$ be an arbitrary SMBS, and consider its corresponding 0-1 labels of the vertices of a hypercube Q_N in the Euclidean space $\mathbb{R}^N$.

Definition 6.18 (Level of Separability). The *level of separability of* $\mathcal{S}$ is the least positive integer d such that there exists positive separator hyperplanes $\pi_1, \ldots, \pi_d$, where all the pathsets of $\mathcal{S}$ lie on the same side as the unit vector for all the hyperplanes, and all the cutsets do not meet the previous condition, at least for one hyperplane.

PROPOSITION 6.19. *Let* $\mathcal{S}$ *be an arbitrary SMBS, and let* $mc = |\mathcal{MC}|$ *be the number of all its mincuts. Therefore, the level of separability* d *verifies* $d \leq mc$.

PROOF. Assume that $x^1, \ldots, x^{mc}$ is the list of all the mincuts of $\mathcal{S}$. Consider the sets $S_i = \{j : x^i_j = 0\}$, that represent the non-operational states for the mincut x^i. Observe that the mincut x^i does not meet the inequality $\pi_i : \sum_{j \in S_i} x_j \geq 1$. Furthermore, the hyperplanes $\pi_1, \ldots, \pi_{mc}$ meet the definition 6.18, and the result follows. □

7 CONCLUDING REMARKS

Topological network design usually deals with deterministic models, while network reliability is intrinsically probabilistic in nature. However, the reliability maximization determines special topologies, that are incidentally related with network optimization models.

In this paper, a direct interplay between the Network Utility Problem (NUP) and a level of separability in Stochastic Binary Systems (SBS) is met. While trees, cycles and arborescences are separable in the all-terminal reliability model, only trees are separable under the source-terminal reliability model. Curiously enough, these separable models define structures with maximum utility in terms of the NUP.

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