Analysing the Complexity of Facility Location Problems with Capacities, Revenues, and Closest Assignments

Christina Büsing*
Combinatorial Optimization
RWTH Aachen University, Germany
buesing@combi.rwth-aachen.de

Timo Gersing†
Combinatorial Optimization
RWTH Aachen University, Germany
gersing@combi.rwth-aachen.de

Sophia Wrede†
Combinatorial Optimization
RWTH Aachen University, Germany
wrede@combi.rwth-aachen.de

ABSTRACT

Facility Location Problems with Capacities, Revenues, and Closest Assignments (FLP-CRCA) are an extension of the well known, strongly $\mathcal{NP}$-complete Facility Location Problem (FLP). With this extension, we recognise that facilities have an upper capacity on the customers to be served, but also need to generate a minimum revenue to be operated economically. Furthermore, we acknowledge that customers have a strong preference towards their closest facility.

We show that finding a feasible solution for FLP-CRCA is already strongly $\mathcal{NP}$-complete if the underlying graph forms a star, but that the problem can be solved efficiently on paths and cycles. In the case where the number of facilities is fixed, we propose a pseudo-polynomial algorithm and show that the problem is weakly $\mathcal{NP}$-complete under this condition. Our results also hold for FLPs with closest assignments and *either* capacities *or* revenues.

1 INTRODUCTION

Facility Location Problems (FLPs) are one of the most fundamental problems in combinatorial optimization [15]. In their most basic version, facilities providing some kind of service for customers need to be opened at potential sites and customers are assigned to these facilities. The aim is to minimize the total cost consisting of opening cost for the facilites and the service cost or traveling cost for assigning the customers. Part of this problem's success is based on the broad applicability to real-world problems [1, 16]. In practice, further constraints need to be considered. Upper capacities on the facilities, i.e., the number of customers or the demand a facility can serve, prevent that customers wait too long at facilities. Such problems are referred to as Capacitated FLPs (CFLPs). One main difference between CFLPs and FLPs is that now customers can not always be served by their closest facility. In reality, however, customers are often free to choose their facility and prefer their closest one. This potentially leads to some facilities being overloaded, while others only serve few customers. In order to prevent such situations, the property of *closest assignments* is demanded: a constraint stating that each customer is to be assigned to their closest open facility; cf. [8, 10] for reviews of closest assignment constraints in integer programming. The closest assignment and upper capacities improve the service quality for the customers. Facilities, however, are often assumed to be operated by individual owners. In order to economically survive, they have to generate a minimum threshold of revenues by serving customer demands. This is achieved by introducing lower bounds on the revenue a facility accumulates. A real world example which can be modeled through Facility Location Problems *with capacities, revenues and closest assignments* is the optimized distribution of pharmacies in a given area under the following assumptions: due to time and space constraints only a limited number of customers can be served; a minimum revenue has to be generated by serving citizens due to the financial independence; citizens use their closest open pharmacy.

The basic FLP is $\mathcal{NP}$-complete in the strong sense [6, 14], which makes the computation of optimal solutions in acceptable time unlikely. This complexity result can be extendend to all of the problem's generalizations. However, if the FLP is defined on a graph, where the assignment costs are equal to the distances between the customer nodes and the facility nodes, the FLP can be solved efficiently if the graph is a tree [6]. For CFLPs, (pseudo-)polynomial algorithms are known in special cases [11, 12]. To the best of the authors' knowledge, the literature on FLPs with Closest Assignments (and Capacities) focuses on strengthening integer programming formulations or developing heuristics [2–4, 7, 8, 10, 13, 17]. Research on the computational complexity of FLPs with Closest Assignments mixed with capacity- and revenue-constraints is still missing.

In this paper, we analyse Facility Location Problems with Capacities, Revenues, and Closest Assignments (FLP-CRCA) and derive some settings that are $\mathcal{NP}$-complete and some that are polynomially solvable. More specifically, we show that:

- finding a feasible solution for the FLP-CRCA is strongly $\mathcal{NP}$-complete on star graphs - contrary to the general FLP, where an optimal solution on trees can be found in polynomial time [6] (Section 3).
- the FLP-CRCA on paths and cycles can be solved efficiently, contrary to the CFLP (Section 4).
- for a fixed number of facilities, there is a pseudo-polynomial algorithm and the problem becomes weakly $\mathcal{NP}$-complete (Section 5).
- these complexity results hold for FLPs with Closest Assignments and *either* capacities *or* revenues.

With this work we close the research gap regarding the computational complexity of Facility Location Problems with Closest Assignments mixed with capacitiy- and revenue-constraints.

*Supported by the German Federal Ministry of Education and Research (grant no. 05M16PAA) within the project "HealthFaCT - Health: Facility Location, Covering and Transport", by the Freigeist-Fellowship of the Volkswagen Stiftung and by the German research council (DFG) Research Training Group 2236 UnRAVeL.
†Supported by the German Federal Ministry of Education and Research (grant no. 05M16PAA) within the project "HealthFaCT - Health: Facility Location, Covering and Transport".

2 PROBLEM DEFINITION AND NOTATION

In this paper, we study the following setting. As an underlying structure consider an undirected graph $G = (V, E)$, where V denotes the set of nodes and E the set of edges. The nodes represent the locations of customers as well as the locations for potential facilities, while the edges represent the street network connecting these sites.

The costs of assigning node $v \in V$ to facility $f \in V$ are denoted by $\tau(v, f) \in \mathbb{Z}_{>0}$ and are defined to be the distance of a shortest (v, f)-path with respect to weights $\delta_e > 0$ on the edges $e \in E$. To each node $v \in V$, we assign parameters, $(r_v, d_v, R_v, C_v, c_v) \in \mathbb{Z}^5_{\geq 0}$, where $r_v \in \mathbb{Z}_{\geq 0}$ represents the *revenue* a customer generates for the serving facility and $d_v \in \mathbb{Z}_{\geq 0}$ represents the customer's *demand* a serving facility has to satisfy. Furthermore, if a facility opens at node v, it has to accumulate a minimum amount of revenue, denoted with $R_v \in \mathbb{Z}_{\geq 0}$ and the aggregated demand must not exceed the capacity $C_v \in \mathbb{Z}_{\geq 0}$. Lastly, parameter c_v denotes the costs of opening a facility at node v.

With these considerations, we define the optimization problem studied here.

Definition 2.1 (FLP-CRCA). We are given an undirected graph with five non-negative parameters for each node and positive weights on the edges, $G = (V, E, (r_v, R_v, d_v, C_v, c_v)_{v \in V}, (\delta_e)_{e \in E})$. Then, the FLP-CRCA consists in finding a subset $F \subseteq V$ of nodes for opening facilities and an assignment $\Lambda : V \longrightarrow F$ of customers to these facilities such that

(1) for each open facility its lower bound on revenue and upper bound on capacity are not violated, i.e., $R_f \leq \sum_{v \in \Lambda^{-1}(f)} r_v$ and $\sum_{v \in \Lambda^{-1}(f)} d_v \leq C_f$ for all $f \in F$,
(2) each customer is assigned to their closest open facility, that is, there exist no $v \in V$ and $f \in F$ with $\tau(v, f) < \tau(v, \Lambda(v))$,
(3) the cost of opening facilities and distances of the customers is minimized, that is $\sum_{f \in F} c_f + \sum_{v \in V} \tau(v, \Lambda(v)) \longrightarrow \min$.

Note that we assume, that the demand of a customer cannot be split between two different facilities.

3 COMPLEXITY OF THE FLP-CRCA

It is common knowledge that the CFLP is strongly $\mathcal{NP}$-complete. However, to emphasize the difference in the complexity of the FLP-CRCA and the CFLP, we first provide a reduction showing that it is already strongly $\mathcal{NP}$-complete to construct any solution for the CFLP at all. This demonstrates that the CFLP's complexity is independent of a potential underlying graph defining the costs – in contrast to the FLP-CRCA, as we will see in Section 4.

THEOREM 3.1. *Finding a feasible solution for CFLP is strongly $\mathcal{NP}$-complete.*

PROOF. We reduce from the strongly $\mathcal{NP}$-complete problem 3-PARTITION [9] to a CFLP-instance. Let $\mathcal{I} = (A, (s_a)_{a \in A}, B)$ be an instance of 3-PARTITION, with A being a set of $3m$ elements, sizes $s_a \in \mathbb{Z}_{\geq 0}$ for each element $a \in A$, such that $s_a \in (B/4, B/2)$ and $\sum_{a \in A} s_a = mB$, for a bound $B \in \mathbb{Z}_{\geq 0}$. The task is to partition A into disjunct subsets $A_1, \ldots, A_m$ such that $\sum_{a \in A_i} s(a) = B$ holds for all $i \in \{1, \ldots, m\}$. We construct from this a CFLP-instance without costs $\mathcal{I}' = (J, (d_j)_{j \in J}, I, (C_i)_{i \in I})$ with customers J, demands d_j, facilities I, and capacities C_i. For each $a \in A$, we define one facility and one customer, that is, $J = I = A$. Furthermore, we set the demands equal to the sizes $d_a = s_a$ and the capacities equal to the bound $C_a = B$.

Note that, there exists a partitioning $A_1, \ldots, A_m$ of A that is a solution to instance $\mathcal{I}$ of 3-PARTITION *iff* there exists a feasible solution to constructed CFLP-instance $\mathcal{I}'$.

CFLP is in $\mathcal{NP}$ since we can test the feasibility of an assignment of customers such that each open facility's capacity constraint is met in linear time w.r.t. the size of set A. □

Considering a lower bound on the revenue is a generalization of CFLPs. The closest assignment condition, however, brings a certain structure to feasible solutions of FLP-CRCA-instances. We will see in the next section that this makes the considered problem easier on certain graph classes. However, already considering trees or stars as underlying networks leads to an $\mathcal{NP}$-complete problem.

THEOREM 3.2. *Finding a feasible solution for FLP-CRCA on stars is strongly $\mathcal{NP}$-complete.*

PROOF. We reduce again from 3-PARTITION, this time to an FLP-CRCA-instance. Let $\mathcal{I} = (A, (s_a)_{a \in A}, B)$ be an instance of 3-PARTITION as defined in the proof of Theorem 3.1. Any such instance will be transformed into an FLP-CRCA-instance $\mathcal{I}' = (V, E, (r_v, R_v, d_v, C_v, c_v)_{v \in V}, (\delta_e)_{e \in E})$ as follows. The set of nodes V contains one node a for every element $a \in A$ and one extra node ξ; hence, $|V| = |A| + 1$. Set $R_a = C_a = B$ and $r_a = d_a = s_a$, for each $a \in V \setminus \{\xi\}$. For node ξ, set $r_\xi = R_\xi = d_\xi = C_\xi = 0$. Next, introduce the set of edges $e \in E$. Connect the nodes so that the underlying graph is an $|A|$-star, where node ξ is the center. Set $\delta_e = 1$ for all edges $e \in E$. Note, due to the choice of parameters in $\mathcal{I}'$, all facilities need to be opened at a leaf in a feasible solution. Due to the underlying star structure, each customer is either indifferent between the facilities or uses the facility located at its own node.

Again, there exists a partitioning $A_1, \ldots, A_m$ of A that is a solution to instance $\mathcal{I}$ of 3-PARTITION *iff* there exists a feasible solution to constructed FLP-CRCA-instance $\mathcal{I}'$.

Finding the nearest facility for a node takes $O(|V|)$ on stars. Hence, the problem is still in $\mathcal{NP}$, as testing whether each customer is served by their nearest open facility can be done in $O(|V|^2)$. □

Thus, no (pseudo-)polynomial algorithm exists, unless $\mathcal{P} = \mathcal{NP}$. However, in the next section, we will see that, in contrast to the CFLP, feasible solutions on paths and cycles can be computed efficiently.

4 POLYNOMIAL SPECIAL CASES

If the underlying graph is either a path or a cycle, any instance of the FLP-CRCA can be solved efficiently. We show this via a reduction to the polynomially solvable SHORTEST S-T-PATH PROBLEM ON DIRECTED ACYCLIC GRAPHS.

4.1 FLP-CRCA on Paths

We first consider the case where the underlying graph of the FLP-CRCA is a path $p = (1, \ldots, n)$. Since we have positive edge weights, paths have the important property that all customers between the most-left and most-right customer of the same facility must be also served by it. That is, for every solution $(F, \Lambda : V \longrightarrow F)$, the set of customers $\Lambda^{-1}(f)$ assigned to a facility $f \in F$ forms an interval $\{l_i, \ldots, u_i\}$ containing f. Hence, the customers can be partitioned into such intervals and every solution can be represented by a sequence of triples $(l_i, f_i, u_i)_{i=1}^k$, where f_i denotes the location of the i-th facility and nodes l_i, u_i the most-left and most-right customer served by it. Note that, for $f_1 < \cdots < f_k$, we have $l_i = u_{i-1} + 1$, with $u_0 := 0$, for all $i \in [k]$. Hence, it suffices to consider tuples (f_i, u_i) for a complete representation of a solution. However, while any feasible solution corresponds to a sequence of tuples $(f_i, u_i)_{i=1}^k$, not every sequence represents a feasible solution.

Definition 4.1. We call $(f, u) \in V^2$ a *feasible tuple* if $f \leq u$. We call a sequence of feasible tuples $(f_i, u_i)_{i=1}^k$ a *feasible sequence* if it meets the following properties:

(a) it holds $u_{i-1} < f_i$ and $u_k = n$,
(b) the customers served by facility f_i yield $\sum_{v=u_{i-1}+1}^{u_i} r_v \geq R_{f_i}$ and $\sum_{v=u_{i-1}+1}^{u_i} d_v \leq C_{f_i}$,
(c) for $i \geq 2$, facility f_i is not closer to customer u_{i-1} than facility f_{i-1} and facility f_{i-1} is not closer to customer $u_{i-1}+1$ than facility f_i, that is, $\tau(u_{i-1}, f_i) \geq \tau(u_{i-1}, f_{i-1})$ as well as $\tau(u_{i-1}+1, f_i) \leq \tau(u_{i-1}+1, f_{i-1})$.

The relation between a feasible sequence and a solution to the FLP-CRCA is elaborated in the next lemma.

Lemma 4.2. *There is a 1-1 correspondence between feasible solutions of the FLP-CRCA on paths and feasible sequences* $(f_i, u_i)_{i=1}^k$.

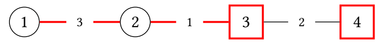
Proof. We already stated above that every solution (F, Λ) can be represented by a sequence $(f_i, u_i)_{i=1}^k$ with $f_1 < \cdots < f_k$ and $\{f_1, \ldots f_k\} = F$ as well as $u_i = \max\{v \in [n] | \Lambda(v) = f_i\}$. This sequence meets Property (a), as otherwise the sets $\Lambda^{-1}(f_i)$ would not form intervals. Furthermore, Properties (b) and (c) follow directly from Properties (1) and (2) in Definition 1.1.

Conversely, every feasible sequence $(f_i, u_i)_{i=1}^k$ defines a solution $F = \{f_1, \ldots f_k\}$ with $\Lambda^{-1}(f_i) = \{u_{i-1}+1, \ldots, u_i\}$. This solution is feasible, as Property (1) in Definition 1.1 follows from Property (b); Property (2) is fulfilled, as no customer in the interval $\Lambda^{-1}(f_i)$ wants to deviate from facility f_i *iff* this is true for the customers at the boundaries, which is ensured by Property (c). □

For finding a feasible solution to the FLP-CRCA, we construct a directed auxiliary graph $G' = (V', A', (w_a)_{a \in A'})$, in which each $s-t$-path corresponds to a feasible sequence. We introduce one node for each feasible tuple together with two extra nodes s, t, i.e., $V' = \{(f, u) \in [n]^2 \mid f \leq u\} \cup \{s, t\}$. Starting at node s, the first arc that we choose on our $s-t$-path corresponds to the first tuple in our sequence. Hence, we have $(s, (f, u)) \in A'$ *iff* tuple (f, u) meets Property (b). Afterwards, choosing an arc from (f', u') to (f, u) corresponds to (f, u) being the successor of (f', u') in our sequence. Hence, this arc exists *iff* Properties (a) – (c) are fulfilled. The last tuple in the sequence needs to meet $u = n$, thus $((f, n), t) \in A'$ for all $f \in [n]$. In conclusion,

$$\begin{aligned} A' = \{&(s,(f,u)) \in \{s\} \times (V' \setminus \{s,t\}) \mid \text{ fulfills (b)}\} \\ \cup \{&((f',u'),(f,u)) \in (V' \setminus \{s,t\})^2 \mid \text{ fulfills (a) – (c)}\} \\ \cup \{&((f,u),t) \in (V' \setminus \{s,t\}) \times \{t\} \mid u = n\}. \end{aligned}$$

FLP-CRCA-instance on a path:



	1	2	3	4
R_v	2	1	5	3
r_v	2	2	2	4
C_v	3	1	6	4
d_v	2	1	2	3
c_v	5	1	6	2

Auxiliary Graph G' :

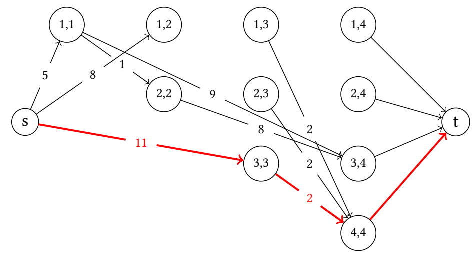


Figure 1: An instance of FLP-CRCA with an optimal solution, opening facilities 3 and 4, of value 13 and the corresponding auxiliary graph G' with a shortest path of the same value.

Finally, we define weights on the arcs such that the cost of an $s-t$-path equals the cost of the corresponding solution:

$$w_a = \begin{cases} 0, & \text{if } a = ((f,u),t) \in A' \\ c_f + \sum_{v=1}^{u} \tau(f,v), & \text{if } a = (s,(f,u)) \in A' \\ c_f + \sum_{v=u'+1}^{u} \tau(f,v), & \text{if } a = ((f',u'),(f,u)) \in A'. \end{cases}$$

For an example of the relationship between an FLP-CRCA-instance on paths and its auxiliary graph, see Figure 1.

The following lemma states that there exists a solution to the FLP-CRCA on paths *iff* there exists an $s-t$-path in the corresponding auxiliary graph G'. In this case, the cost of an optimal solution is equal to the cost of a shortest $s-t$-path.

Lemma 4.3. *There is a cost-preserving 1-1 correspondence between solutions of the FLP-CRCA and $s-t$-paths in G'.*

Proof. By construction of auxiliary graph G', every $s-t$-path $p' = (s, (f_1, u_1), \ldots, (f_k, n), t)$ corresponds to exactly one feasible sequence $(f_i, u_i)_{i=1}^k$, and thus, due to Lemma 4.2, also to one solution (F, Λ) with $F = \{f_1, \ldots f_k\}$ and $\Lambda^{-1}(f_i) = \{u_{i-1}+1, \ldots u_i\}$.

For the cost of p', it holds

$$\begin{aligned}\sum_{a\in p'} w_a &= w_{(s,(f_1,u_1))} + \sum_{i=2}^{k} w_{((f_{i-1},u_{i-1}),(f_i,u_i))} + w_{((f_k,n),t)}\\ &= c_{f_1} + \sum_{v=1}^{u_1} \tau(f_1,v) + \sum_{i=2}^{k}(c_{f_i} + \sum_{v=u_{i-1}+1}^{u_i} \tau(f_i,v))\\ &= \sum_{f\in F}(c_f + \sum_{v\in\Lambda^{-1}(f)} \tau(f,v))\\ &= \sum_{f\in F} c_f + \sum_{v\in V} \tau(v,\Lambda(v)),\end{aligned}$$

which is exactly the cost of solution (F, Λ). □

Checking whether an arc is feasible and computing its weight takes $O(n)$ steps for each of the $O(n^4)$ possible arcs. Hence, we can transform any FLP-CRCA instance on paths in such an auxiliary graph in $O(n^5)$ steps. For computing a shortest $s-t$-path, first note that G' is an acyclic graph. This can be seen by sorting the nodes in lexicographic order, i.e., $s, (1,1), (1,2), \dots, (1,n), (2,2), \dots, (n,n), t$, and recognizing that there exists no backward-arc due to (a). Since single-source shortest paths in directed acyclic graphs can be computed in $O(|V'| + |A'|)$ [5] , we can compute a shortest path in $O(n^4)$ steps. Altogether, we obtain the following.

Theorem 4.4. *An optimal solution to FLP-CRCA can be computed in $O(n^5)$ steps.*

In practice, constructing the graph and finding a shortest path can be sped up significantly by doing both in parallel. For this, we iterate over the nodes in the lexicographic ordering. When reaching node (f,u), we only check whether there exist outgoing arcs if node (f,u) was reached before, as those arcs do not belong to a shortest path otherwise.

From a modeler's perspective, this result is useful if the real world street network can be modeled as a path. For example, when deciding where to open pharmacies in rural areas, where multiple villages are connected by the same road, and each village is too small such that at least one pharmacy could economically operate in each of them. Real world examples are mountain passes with hotels or small villages along the road in Switzerland.

Besides the practical applicability, it is also interesting to analyse the theoretical impact of the constraints on the computational complexity on basic graph structures. In the next subsection, for example, we study how the algorithm on paths can be extended to cycles.

4.2 FLP-CRCA on Cycles

In order to solve the FLP-CRCA on cycles, we reduce it to multiple subproblems that are similar to the FLP-CRCA on paths.

Consider an instance on a cycle $G = (V,E)$ and an optimal solution $(F^*, \Lambda^* : V \to F^*)$. Note that there exists an edge $e^* \in E$ such that the above solution is also feasible for the FLP-CRCA on the path $G_{e^*} = (V, E\backslash\{e^*\})$. To see this, consider a forest within G that connects each customer $v \in V$ with its serving facility $\Lambda^*(v)$ via a shortest path. The missing edges in the forest are exactly those that can be deleted from E.

For now, assume that we are given e^*. Additionally, assume w.l.o.g. $e^* = \{n,1\}$, that is, G_{e^*} is the path $p_{e^*} = (1,\dots,n)$. In the best case, we can compute a shortest path in the auxiliary graph G'_{e^*}, as defined in the previous subsection, that corresponds to (F^*, Λ^*). However, it is possible that the computed shortest path in G'_{e^*} corresponds to a solution $(F', \Lambda' : V \to F')$ that is not feasible for the original instance on G, as customer n might prefer facility $\Lambda'(1)$ over $\Lambda'(n)$ or customer 1 might prefer facility $\Lambda'(n)$ over $\Lambda'(1)$. Hence, we want to restrict ourselves to paths in G'_{e^*} that can be "glued together" at the first tuple (f_1,u_1) and the last tuple (f_k,n) such that the corresponding solution is feasible on G. For this, we require that (f_1,u_1) can be a successor of (f_k,n) in accordance to Definition 4.1. Unfortunately, this yields a shortest path problem in which the feasible paths depend on the first chosen arc $(s,(f_1,u_1))$, which is usually problematic. We resolve this issue by fixing a potential last arc $((f_k,n),t)$ and removing all arcs $(s,(f_1,u_1))$ from G'_{e^*} for which (f_1,u_1) is not a feasible successor of (f_k,n). Then every $s-(f_k,n)$-path in the resulting graph corresponds to a solution that is feasible to the original problem. Moreover, the path corresponding to the optimal solution (F^*, Λ^*) is contained in the graph resulting from fixing arc $((\Lambda^*(n),n),t)$ and can thus be found by computing a shortest $s-(\Lambda^*(n),n)$-path.

Naturally, we neither know $\Lambda^*(n)$, nor do we have e^* in advance. However, testing all possible combinations $e^* \in E$ and $\Lambda^*(n) \in V$, we are guaranteed to find one yielding an optimal solution. In summary, we solve n^2 shortest path problems, each requiring $O(n^4)$ steps. Deleting arcs $(s,(f_1,u_1))$ from G'_e for a fixed last arc $((f_k,n),t)$ can be done on the fly in constant time for each arc while computing the shortest path. As stated in the previous subsection, computing an auxiliary graph G'_e requires $O(n^5)$ steps. Note that $G'_{\{i,i+1\}}$ can be computed efficiently from $G'_{\{i-1,i\}}$ by reusing most of the graph. This leads us to the following statement.

Theorem 4.5. *The FLP-CRCA on cycles can be solved in $O(n^6)$.*

For a better understanding of the procedure for solving FLP-CRCA-instances on cycles, consider the following example.

Example 4.6. Consider an FLP-CRCA instance on a cycle with parameters as introduced in Figure 1; cf. Figure 2. When fixing arc $((3,4),t)$, arc $(s,(3,3))$ has to be removed since tuple $(3,3)$ is not a feasible successor of $(3,4)$; cf. auxiliary graph $G'_{\{4,1\}}$ in Figure 2. An optimal solution of value 14 can be achieved by opening facilities 1 and 4. Note that, arc $(s,(3,3))$ has also to be removed when fixing arc $((4,4),t)$: otherwise, customer 1 would deviate from facility 3 to facility 4 in the underlying cycle.

5 FIXED NUMBER OF FACILITIES

If the number of open facilities $k \in \mathbb{N}$ is fixed, we refer to the problem as the k-FLP-CRCA. That is, we consider instances of the FLP-CRCA and are only interested in optimal solutions where k facilities are opened and k is not part of the input.

Lemma 5.1. *Finding a feasible solution for k-FLP-CRCA on stars is at least weakly $\mathcal{NP}$-complete, already for $k = 2$.*

Proof. The claim can be seen by a reduction from the weakly $\mathcal{NP}$-complete problem Partition [9]. In Partition, a finite set A is considered; each element $a \in A$ has a weight, $s_a \in \mathbb{Z}^+$. The question

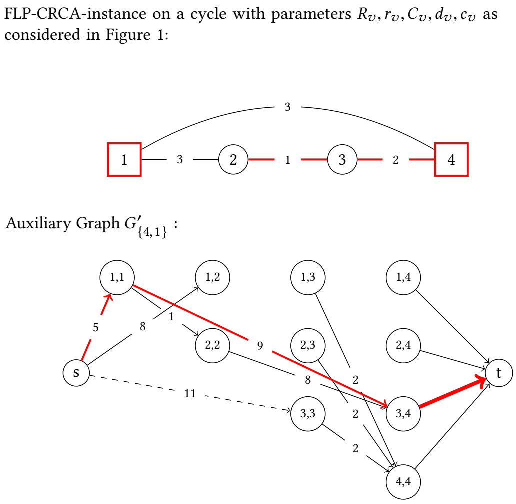


Figure 2: An instance of FLP-CRCA on a cycle with deleted edge $e^* = \{4, 1\}$. Note that, the optimal solution found in Figure 1 is not feasible here.

is whether A can be partitioned into two disjunct sets $A', A \setminus A'$ such that $\sum_{a \in A'} s_a = \sum_{a \in A \setminus A'} s_a =: B$.

For a given Partition instance $\mathcal{I} = (A, (s_a)_{a \in A})$, we construct a k-FLP-CRCA instance with $k = 2$ on a star analogously to the proof of Theorem 3.2. We introduce one leaf node for each element $a \in A$ and one center node ξ. We set $d_a = r_a = s_a$ and $C_a = R_a = B$, for $a \in V \setminus \{\xi\}$ as well as $d_\xi = C_\xi = r_\xi = R_\xi = 0$. Again, we define $\delta_e = 1$ for all edges $e \in E$ of the star

Note that, there exists a partitioning $A', A \setminus A'$ of A that is a solution to instance $\mathcal{I}$ of Partition *iff* there exists a feasible solution to constructed k-FLP-CRCA-instance $\mathcal{I}'$.

As a special case of FLP-CRCA, the problem is in $\mathcal{NP}$. □

In the following, we introduce a dynamic program that solves any k-FLP-CRCA-instance in pseudo-polynomial time, thus proving that the problem is indeed weakly $\mathcal{NP}$-complete. To that end, we fix a set of facilities and test in pseudo-polynomial time whether we find a closest assignment satisfying the revenue- and demand-constraints. Let $F \in \binom{V}{k}$ be a fixed set of facilities. We first determine for each customer $v \in V$ the set of closest facilities $F_v \subseteq F$ to which v can be assigned to. Afterwards, we compute labels

$$b : \{1, \dots, |V|\} \times \bigtimes_{f \in F} \left(\{0, \dots, C_f\} \times \{0, \dots, R_f\}\right) \to \mathbb{R} \cup \{\infty\},$$

where $b(i, C'_{f_1}, R'_{f_1}, \dots, C'_{f_k}, R'_{f_k})$ states the minimum cost of assigning customers $\{1, \dots, i\} \subseteq V$ so that facility $f \in F$ accumulates demand of at most C'_f and revenue of at least R'_f, that is the value of the subproblem

$$\min_{\Lambda:[i]\to F} \left\{ \sum_{f \in F} c_f + \sum_{v \in [i]} \tau(v, \Lambda(v)) \;\middle|\; \begin{array}{rl} \Lambda(v) \in F_v & \forall v \in [i] \\ \sum_{v \in \Lambda^{-1}(f)} d_v \leq C'_f & \forall f \in F \\ \sum_{v \in \Lambda^{-1}(f)} r_v \geq R'_f & \forall f \in F \end{array} \right\}. \quad (1)$$

Note that, the labels with finite cost and parameters $i = |V|$, $C'_f \in \{0, 1, \dots, C_f\}$ and $R' = R_f$ for all $f \in F$ are exactly the labels corresponding to feasible solutions in (1) of considered k-FLP-CRCA-instance with fixed set of facilities F; the cost of an optimal solution for fixed F is $b(|V|, C_{f_1}, R_{f_1}, \dots, C_{f_k}, R_{f_k})$.

To compute the labels, we use the recursion formula

$$\begin{aligned} & b(i, C'_{f_1}, R'_{f_1}, \dots, C'_{f_k}, R'_{f_k}) \\ & = \min_{f_j \in F_i} \{ b(i-1, \dots, C'_{f_j} - d_i, R'_{f_j} - r_i, \dots, C'_{f_k}, R'_{f_k}) + \tau(i, f_j) \} \end{aligned}$$

together with the base values

$$b(0, C'_{f_1}, R'_{f_1}, \dots, C'_{f_k}, R'_{f_k}) = \begin{cases} \sum_{f \in F} c_f, & \text{if } C'_f \geq 0,\ R'_f \leq 0\ \forall f \in F \\ \infty, & \text{otherwise.} \end{cases}$$

This leads to the following result.

Theorem 5.2. *The k-FLP-CRCA is weakly $\mathcal{NP}$-complete and can be solved in $O(\binom{|V|}{k} \cdot k \cdot |V| \cdot \prod_{f \in F}(C_f \cdot R_f))$ steps by using the above dynamic program.*

Proof. We show that the values computed by the recursion formula are equal to the values of the subproblems (1).

For the base values, assigning no customers leaves us with the opening costs $\sum_{f \in F} c_f$. Furthermore, assigning no customers meets the capacity- and revenue-constraints of subproblem (1) *iff* $C'_f \geq 0$ and $R'_f \leq 0$ holds for all $f \in F$.

When assigning customer i to a fixed facility $f' \in F_i$, the minimum cost for assigning $[i]$ and respecting capacities and revenues $(C'_f, R'_f)_{f \in F}$ in subproblem (1) is given by

$$\tau(i, f') + \min_{\Lambda \in \mathcal{P}} \left\{ \sum_{f \in F} c_f + \sum_{v \in [i-1]} \tau(v, \Lambda(v)) \right\},$$

with

$$\mathcal{P} = \left\{ \Lambda : [i-1] \to F \;\middle|\; \begin{array}{rl} \Lambda(v) \in F_v & \forall v \in [i-1] \\ \sum_{v \in \Lambda^{-1}(f)} d_v \leq C'_f & \forall f \in F \setminus \{f'\} \\ \sum_{v \in \Lambda^{-1}(f')} d_v \leq C'_{f'} - d_i & \\ \sum_{v \in \Lambda^{-1}(f)} r_v \geq R'_f & \forall f \in F \setminus \{f'\} \\ \sum_{v \in \Lambda^{-1}(f')} r_v \geq R'_{f'} - r_i & \end{array} \right\},$$

which is by induction exactly the value of

$$b(i-1, \dots, C'_{f'} - d_i, R'_{f'} - r_i, \dots, C'_{f_k}, R'_{f_k}) + \tau(i, f').$$

Taking the minimum cost over all $f' \in F_i$ yields the optimal assignment, which proves the correctness of the recursion formula.

For each set of facilities $F \in \binom{V}{k}$, computing the closest facilities $F_i \subseteq F$ for a customer $i \in V$ can be done in $O(k)$ steps if we compute all distances in a preprocessing step. Computing a label requires comparing $|F_i| \leq k$ values and can thus be done in $O(k)$ steps. For every set of facilities $F \in \binom{V}{k}$, we compute $|V| \cdot \prod_{f \in F}(C_f \cdot R_f)$ many labels, which yields the time-complexity stated above.

The weak $\mathcal{NP}$-completeness of the k-FLP-CRCA follows together with Lemma 5.1. □

Thus, as in the general Facility Location Problem, the FLP-CRCA is easier to solve if the number of open facilities is fixed. Note that this result is independent of the underlying network.

6 CONCLUSION

We showed that Facility Location Problems with Capacities, Revenue, and Closest Assignments (FLP-CRCA) are already computationally intractable if the underlying graph forms a star - contrary to the famous Facility Location Problem, which is known to be tractable on trees [6]. On paths and cycles, however, the FLP-CRCA turns out to be computationally tractable - unlike Capacitated Facility Location Problems. This discrepancy is caused by the closest assignment property, which brings a special structure to the solutions for FLP-CRCA-instances. Furthermore, we showed that, if the number of open facilities is fixed, the FLP-CRCA is weakly $\mathcal{NP}$-complete for any underlying graph class. All results presented here also hold for Facility Location Problems with Closest Assignments and *either* capacities *or* revenues.

Further work includes the study of approximation algorithms and analysing the complexity of further graph classes.

ACKNOWLEDGMENTS

The authors would like to thank the anonymous referees for their constructive feedback.

REFERENCES

[1] Amir Ahmadi-Javid, Pardis Seyedi, and Siddhartha S Syam. 2017. A survey of healthcare facility location. *Computers & Operations Research* 79 (2017), 223–263. https://doi.org/10.1016/j.cor.2016.05.018

[2] Herminia I. Calvete, Carmen Galé, José A Iranzo, José-Fernando Camacho-Vallejo, and Martha-Selene Casas-Ramírez. 2020. A matheuristic for solving the bilevel approach of the facility location problem with cardinality constraints and preferences. *Computers & Operations Research* 124 (2020), 105066. https://doi.org/10.1016/j.cor.2020.105066

[3] Massimiliano Caramia and Renato Mari. 2016. A decomposition approach to solve a bilevel capacitated facility location problem with equity constraints. *Optimization Letters* 10 (2016), 997–1019. https://doi.org/10.1007/s11590-015-0918-z

[4] Martha-Selene Casas-Ramírez, José-Fernando Camacho-Vallejo, and Iris-Abril Martínez-Salazar. 2018. Approximating solutions to a bilevel capacitated facility location problem with customer's patronization toward a list of preferences. *Appl. Math. Comput.* 319 (2018), 369–386. https://doi.org/10.1016/j.amc.2017.03.051 Recent Advances in Computing.

[5] Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest, and Clifford Stein. 2009. *Introduction to Algorithms.* MIT Press.

[6] Gérard Cornuèjols, George L. Nemhauser, and Laurence A. Wolsey. 1990. The uncapacitated facility location problem. *Discrete Location Theory* (1990). url: https://pdfs.semanticscholar.org/1c16/97671bc4a9a85e8619bc0c8d3e40c1bd27cb.pdf.

[7] Lázaro Cánovas, Sergio García, Martine Labbé, and Alfredo Marín. 2007. A strengthened formulation for the simple plant location problem with order. *Operations Research Letters* 35, 2 (2007), 141–150. https://doi.org/10.1016/j.orl.2006.01.012

[8] Inmaculada Espejo, Alfredo Marín, and Antonio M. Rodríguez-Chía. 2012. Closest assignment constraints in discrete location problems. *European Journal of Operational Research* 219, 1 (2012), 49–58. https://doi.org/10.1016/j.ejor.2011.12.002

[9] Michael R. Garey and David S. Johnson. 1979. *Computers and Intractability: A Guide to the Theory of $\mathcal{NP}$-completeness.* W. H. Freeman & Co., New York, USA.

[10] Ross A Gerrard and Richard L Church. 1996. Closest assignment constraints and location models: Properties and structure. *Location Science* 4, 4 (1996), 251–270. https://doi.org/10.1016/S0966-8349(97)00001-6

[11] Edward Kh Gimadi and Anna A. Kurochkina. 2019. Time complexity of the ageev's algorithm to solve the uniform hard capacities facility location problem. In *Optimization and Applications - 9th International Conference, OPTIMA 2018, Revised Selected Papers (Communications in Computer and Information Science)*, Yury Kochetov, Michael Khachay, Yury Evtushenko, Vlasta Malkova, Mikhail Posypkin, and Milojica Jacimovic (Eds.). Springer-Verlag GmbH and Co. KG, Germany, 123–130. https://doi.org/10.1007/978-3-030-10934-9_9 9th International Conference on Optimization and Applications, OPTIMA 2018 ; Conference date: 01-10-2018 Through 05-10-2018.

[12] Edward Gimadi, Alexandr Shtepa, and Oxana Tsidulko. 2019. Improved Exact Algorithm for the Capacitated Facility Location Problem on a Line Graph. In *2019 15th International Asian School-Seminar Optimization Problems of Complex Systems (OPCS).* 53–57. https://doi.org/10.1109/OPCS.2019.8880248

[13] Pierre Hanjoul and Dominique Peeters. 1987. A facility location problem with clients' preference orderings. *Regional Science and Urban Economics* 17, 3 (1987), 451–473. https://doi.org/10.1016/0166-0462(87)90011-1

[14] Jakob Krarup and Peter Marc Pruzan. 1983. The simple plant location problem: Survey and synthesis. *European Journal of Operational Research* 12, 1 (1983), 36–81. https://doi.org/10.1016/0377-2217(83)90181-9

[15] Gilbert Laporte, Stefan Nickel, and Francisco Saldanha da Gama (Eds.). 2019. *Location Science.* Springer. https://doi.org/10.1007/978-3-030-32177-2

[16] Derya C Turkoglu and Mujde E Genevois. 2019. A comparative survey of service facility location problems. *Annals of Operations Research* 292 (2019), 399–468. Issue 1. https://doi.org/10.1007/s10479-019-03385-x.

[17] Igor Vasilyev, Xenia Klimentova, and Maurizio Boccia. 2013. Polyhedral study of simple plant location problem with order. *Oper. Res. Lett.* 41 (2013), 153–158. https://doi.org/10.1016/j.orl.2012.12.006