Short Paper

The Load Minimization Problem on cycles

Mariana Escalante
Paola Tolomei
mariana@fceia.unr.edu.ar
ptolomei@fceia.unr.edu.ar
FCEIA - Universidad Nacional de Rosario and CONICET
Rosario, Santa Fe, Argentina

Martín Matamala
Iván Rapaport
mar.mat.vas@dim.uchile.cl
rapaport@dim.uchile.cl
DIM-CMM(ILR-CNRS-2807)
Universidad de Chile
Santiago de Chile, Chile

Luis Miguel Torres
luis.torres@epn.edu.ec
Centro de Modelización Matemática ModeMat
Escuela Politécnica Nacional
Quito, Ecuador

ABSTRACT

In this work we study the Load Minimization Problem in undirected weighted cycles. In this problem, we are given a cycle and a set of weighted origin-destination pairs. The goal is to route all the pairs minimizing the load of the routing according to the given weights. We prove that the problem is NP-complete and that it is 2-approximable. For unitary weights we present a FPT algorithm whose parameter is a natural lower bound for the value of the load.

KEYWORDS

network routing, approximation algorithms, routing and assignment.

1 INTRODUCTION

In the Load Minimization Problem, we are given an undirected graph $G = (V, E)$ and a set of weighted origin-destination pairs $\mathcal{D}$, with weights $\mathbf{w}$; the aim is to find a minimum load routing of these pairs. The load of a routing is the maximum weight of a set of pairwise intersecting routes.

In full generality, the Load Minimization Problem is NP-hard, as deciding whether there is a routing with load equal to zero for a given set of origin-destination pairs on a graph corresponds to the Edge Disjoint Path (EDP) problem, a classical NP-complete problem [9], which remains NP-complete even when restricted to planar graphs [12]. On the other hand, EDP is Fixed Parameter Tractable on the number of demands, i.e., of origin-destination pairs ([14],[10]).

The optimization version of EDP where the number of demands whose routes form a pairwise edge disjoint set is maximized, has an $O(\sqrt{n})$-approximation algorithm [1].

Our work is also related to the Congestion Minimization Problem where a routing is sought which minimizes the number of paths passing through any edge. This problem has an approximation factor of $O(\log n/\log\log n)$ which is matched by an $\Omega(\log n/\log\log n)$-hardness result [3]. When the Congestion Minimization Problem is restricted to cycles it is also known as the Load Routing Problem first discussed in [1] where it was proved to be NP-hard and 2-approximable. This result was improved latter in [11], where the existence of a PTAS (Polynomial Time Approximation Scheme) was proved. It is also known that the problem can be solved in polynomial time when all the weights are the same [5].

Since any set of routes passing through an edge is a set of pairwise intersecting routes, the optimal value of the Load Minimization Problem is an upper bound for the Congestion Minimization Problem. This implies the lower bound $\Omega(\log^{1/2-\epsilon} n)$ for the approximability of the Load Minimization Problem. Nonetheless, neither the PTAS for the Congestion Minimization Problem in cycles nor the polynomial time algorithm when all the demands have the same weight can be extended to the Load Minimization Problem.

A version of the Load Minimization Problem with colors arises in the context of routing in optical networks. The Routing and Wavelength Assignment (RWA) Problem asks to find, for each origin-destination pair, a route through the network and a wavelength or color so that routes with the same wavelength share no edges, and the number of used wavelengths is minimized [13]. A more general problem, the Routing and Spectrum Assignment (RSA) Problem consists in finding, for each origin-destination pair with weight w, a route through the network and a channel of w consecutive frequency slots so that the channels of two demands are disjoint whenever their routes have an edge in common, and the number of used frequency slots is minimized [16]. Naturally, the RWA and RSA problems have been shown to be NP-hard [4, 17]. We note that the RSA problem remains hard when the network is a path [15], whereas the RWA problem is polynomial solvable in this case [6].

In this work we study the complexity of the Load Minimization Problem restricted to cycles. Our first result shows that the Load Minimization Problem in cycles is NP-hard. This result is complemented with a 2-approximation algorithm and with a FPT algorithm when all the demands have the same weight.

2 COMPLEXITY AND APPROXIMATION RESULTS

In order to state and prove our results, we introduce some definitions and notation. We assume that the graph G is a cycle with node set $\{1, 2 \ldots, n\}$ and a fixed orientation of the edges of G in a way that they induce a directed circuit.

For any two nodes i and j of G we denote by $[i, j]$ the unique directed path from i to j in this circuit. Moreover, let $\overline{[i, j]} = [j, i]$; whence, each route in G is given by $[i, j]$, for some $i, j \in \{1, \ldots, n\}$.

Since we are working in an undirected setting, we can assume that each demand is a pair $p = (i, j)$, with $1 \leq i < j \leq n$. A demand $p = (i, j)$ can be routed in two ways: $p^+ = [i, j]$ and $p^- = [j, i]$. The integers i and j are called the ends of p, and $e(p) = \{i, j\}$ is the set of ends of p. Hence, a routing R consists of assigning either p^+ or p^- to each demand p. For a routing R and a demand p, let R_p be the route assigned to p by R.

10.48786/alioeuro.2022.17

For a set of demands $\mathcal{D}' \subseteq \mathcal{D}$, we set $\mathbf{w}(\mathcal{D}') = \sum_{p \in \mathcal{D}'} \mathbf{w}_p$; for a set of routes S of a routing R of $\mathcal{D}$ we set $\mathbf{w}(S) = \mathbf{w}(\{p : R_p \in S\})$.

To get a better understanding of the problem it is convenient to associate with each routing R, the edge-intersection graph of its routes, denoted by H_R. When G is a cycle, H_R is a circular-arc graph and the load of R is the maximum weight, with respect to the weights $\mathbf{w}$ of the demands in $\mathcal{D}$, of a clique in H_R, denoted by $\ell(H_R, \mathbf{w})$, which can be computed in time $O(|\mathcal{D}|^3)$, by Hsu's result ([8]).

Using previous notation, the Load Minimization Problem consists in determining

$$\ell(G, \mathcal{D}, \mathbf{w}) = \min\{\ell(H_R, \mathbf{w}) \mid R \text{ routing of } \mathcal{D}\}.$$

We start by assessing the complexity of the Load Minimization Problem.

Theorem 2.1. *The Load Minimization Problem is NP-hard.*

Proof. The decision problem associated with the Load Minimization Problem is in NP since, given a routing one can check in polynomial time (see above) if its load is less than a given value k.

Now, we prove the completeness by a polynomial reduction from Partition [7]. Let $S = \{x_1, \ldots, x_r\}$ be a multiset of positive integers, an instance of Partition. We have to decide if there is $T \subseteq S$ such that

$$\sum_{x \in T} x = \sum_{x \in S \setminus T} x. \qquad (1)$$

We may assume that $\sum_{r=1}^{n} x_r$ is an even number without loss of generality. Let $V = \{1, 2, 3\}$ be and let $\mathcal{D}$ be composed by r demands $p_i = (1, 2)$ with $w_{p_i} = x_i$ for all $i \in \{1, \ldots, r\}$, and $k = \frac{1}{2} \sum_{r=1}^{n} x_r$. For a given routing R, let R_p be the route assigned to p and let

$$\mathcal{D}_+(R) \doteq \{p \in \mathcal{D} : R_p = p^+\}$$

and

$$\mathcal{D}_-(R) \doteq \{p \in \mathcal{D} : R_p = p^-\}.$$

Suppose that $\ell(G, \mathcal{D}, \mathbf{w}) \leq k$. Then, there is a routing R such that $\ell(H_R, \mathbf{w}) \leq k$. Therefore, $\mathbf{w}(\mathcal{D}_+(R)) = \mathbf{w}(\mathcal{D}_-(R)) = k$. The, the set $T = \{x_i : p_i \in \mathcal{D}_+(R)\}$ satisfies (1) since $S \setminus T = \{x_i : p_i \in \mathcal{D}_-(R)\}$.

Conversely, if there is some T satisfying (1), then the routing R given by $R_{p_i} = p_i^+$, for all i such that $x_i \in T$, and $R_{p_i} = p_i^-$, otherwise has load at most k. Therefore, $k \geq \ell(G, \mathcal{D}, \mathbf{w})$. □

We now prove that the Load Minimization Problem is 2- approximable. For this purpose we need some more definitions.

Definition 2.2. For given demands $p, q \in \mathcal{D}$ we say that p and q *cross* if $|e(p) \cap V(q^+)| = |e(p) \cap V(q^-)| = 1$. When p and q do not cross we say that they are *parallel*.

Then, two demands p and q cross if p has exactly one end in each of the possible routes for q, i.e., in q^+ and q^-. Notice that this implies that q also has exactly one end in each of the routes p^+ and p^-. Moreover, any route of p and any route of q have at least one arc in common.

When drawing the directed cycle like a circle in the plane, a demand can be represented by a chord between its origin and its destination. In this representation chords associated with two demands that cross, intersect in the interior of the circle. Conversely, if the chords do not intersect in the interior of the circle, then the demands are parallel. In Figure 1, $(1, 5)$, $(4, 8)$

$$\mathcal{D} = \{(1, 5), (2, 3), (3, 6), (4, 5), (4, 8), (6, 8)\}$$

$$R = \{[3, 2], [4, 5], [8, 6], [4, 8], [5, 1], [6, 3]\}$$

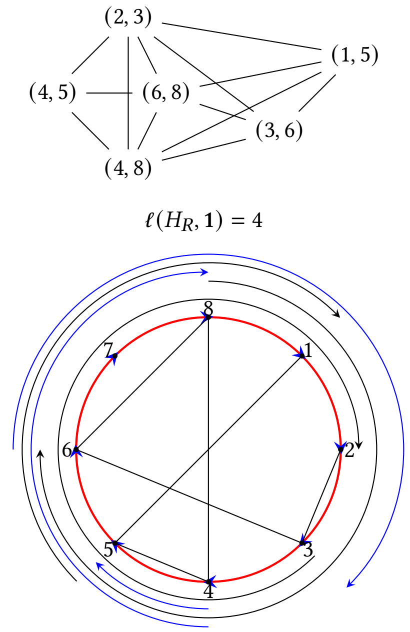

Figure 1: An instance with a feasible solution for w = 1.

and $(3, 6)$ are pairwise crossing, while $(1, 5)$, $(2, 3)$ and $(6, 8)$ are pairwise parallel.

Definition 2.3. Let $\mathcal{D}^+ \subseteq \mathcal{D}$ be the set of demands p such that $|V(p^-)| = |V(p^+)|$.

In the example given in Figure 1, $\mathcal{D}^+ = \{(4, 8), (1, 5)\}$. Notice that $p, q \in \mathcal{D}^+$ cross whenever $e(p) \neq e(q)$. Moreover, for $p \notin \mathcal{D}^+$ either $|V(p^+)| < |V(p^-)|$ or $|V(p^+)| > |V(p^-)|$. Also, as $|V(p^-)| = n - |V(p^+)| + 2$ for any demand p, we have that, if n is even, $\mathcal{D}^+ = \{p \in \mathcal{D} : |V(p^+)| = n/2 + 1\}$ and, if n is odd, $\mathcal{D}^+ = \emptyset$.

We now present Algorithm 1 which assigns routes to $\mathcal{D}^+$ by alternating senses of directions, and then the remaining demands, where the shortest path is chosen.

Algorithm 1 Approximation algorithm for $\ell(G, \mathcal{D}, \mathbf{w})$.

Input: A cycle $G = (V, E)$, a set of demands $\mathcal{D}$ and weights $\mathbf{w} \in \mathbb{Z}_+^{\mathcal{D}}$.
Output: A feasible solution R and $\ell(H_R, \mathbf{w})$.

Compute $\mathcal{D}^+ = \{p_1, p_2, \ldots, p_t\}$ so that demands are listed in lexicographic order
for $k = 1, \ldots, t$ **do**
  if k is odd **then** assign $R_{p_k} = p_k^+$ **else** assign $R_{p_k} = p_k^-$
end for
for $p \in \mathcal{D} \setminus \mathcal{D}^+$ **do**
  if $|V(p^+)| < |V(p^-)|$ **then** assign $R_p = p^+$ **else** assign $R_p = p^-$
end for
Find a clique Q in H_R with $\mathbf{w}(Q)$ maximum.

Our main theoretical contribution is to prove that Algorithm 1 is a 2-approximation algorithm for the Load Minimization Problem in cycles.

It is easy to see that for a routing R, two demands p and q which cross are adjacent in H_R. Indeed, as observed above, $R_p \in \{p^+, p^-\}$ and $R_q \in \{q^+, q^-\}$ share at least one common arc. Then, if $\mathcal{D}'$ is a set of pairwise crossing demands, $\ell(H_R, \mathbf{w}) \geq \mathbf{w}(\mathcal{D}')$ must hold, and the following lower bound for $\ell(G, \mathcal{D}, \mathbf{w})$ is obtained.

Lemma 2.4. *Let G be a cycle and let $\mathcal{D}'$ be a subset of $\mathcal{D}$ such that any two demands in $\mathcal{D}'$ cross. Then, $\ell(G, \mathcal{D}, \mathbf{w}) \geq \mathbf{w}(\mathcal{D}')$.*

Definition 2.5. For two distinct arcs $a = (u, v)$ and $b = (s, t)$ of the directed cycle, let $CUT(a, b)$ be the set of demands p such that

$$|e(p) \cap V([v, s])| = |e(p) \cap V([t, u])| = 1.$$

Hence, demands in $CUT(a, b)$ have one end in $[v, s]$ and the other end in $[t, u]$. Notice that, if $p \in CUT(a, b)$, any route for p must contain exactly one of the arcs a or b, i.e., the sets $\{a, b\} \cap A(p^+)$ and $\{a, b\} \cap A(p^-)$ are both non-empty. This observation leads to the following lower bound for $\ell(G, \mathcal{D}, \mathbf{w})$.

Lemma 2.6. *Let G be a cycle and let $\mathcal{D}$ be a set of demands. Then, for any two distinct arcs a and b, $\mathbf{w}(CUT(a, b)) \leq 2\ell(G, \mathcal{D}, \mathbf{w})$.*

Proof. Let R be a routing of $\mathcal{D}$ and $p \in CUT(a, b)$. As we have seen above, R_p contains either a or b. Thus, $CUT(a, b)$ can be partitioned into two sets $\mathcal{D}_a$ and $\mathcal{D}_b$ of demands whose assigned routes contain a or b, respectively. Each of these sets of demands induces a clique in H_R, as the corresponding routes share (at least) one arc. Hence,

$$\mathbf{w}(CUT(a, b)) = \mathbf{w}(\mathcal{D}_a) + \mathbf{w}(\mathcal{D}_b) \leq 2\ell(H_R, \mathbf{w}).$$

□

Theorem 2.7. *The Load Minimization Problem restricted to cycles is 2-approximable.*

Proof. As we already discused, the clique Q obtained by Algorithm 1 can be computed in time $O(|\mathcal{D}|^3)$.

Let us first prove that Q is a subset of $CUT(a, \hat{a})$, where $\hat{a}$ is the arc opposite to a in G, for some arc a. From the definition of Algorithm 1, for each $p \in \mathcal{D}$ and each arc a of the cycle, if $a \in A(R_p)$, then $\hat{a} \notin A(R_p)$. Let Q_a denote the set of demands in Q whose associated routes contain the arc a, and let a_0 be such that Q_{a_0} is the largest set among all Q_a. We shall prove that $Q \subseteq CUT(a_0, \hat{a_0})$. This is immediate when $Q = Q_{a_0}$, by the definition of Algorithm 1. We claim that for each $p \in Q \setminus Q_{a_0}$ we have that $\hat{a_0} \in A(R_p)$. If this were not the case, then R_p would be contained in one of the two paths obtained from G when we remove a_0 and $\hat{a_0}$.

Let b be the arc in R_p closest to a_0. Then, for each $q \in Q_{a_0}$ we have that $b \in A(R_q) \cap A(R_p)$ which shows that $Q_{a_0} \subseteq Q_b$. However, since $p \in Q_b \setminus Q_{a_0}$ we get a contradiction with the choice of a_0.

After the result in Lemma 2.6 and the fact that $Q \subseteq CUT(a, \hat{a})$, we get that $\mathbf{w}(Q) \leq 2\ell(G, \mathcal{D}, \mathbf{w}))$. This implies that it is an 2-approximation algorithm for the Load Minimization Problem. □

It is worth to see that we can not improve the factor 2 in the algorithm. In fact, consider the following family of instances. For $r \geq 1$, let G be a cycle of length $8r - 1$ and for each $i \in \{1, \ldots, 2r\}$, let $p_i = (4r - i, 4r + i - 1)$. Then, for each $i \in \{1, \ldots, 2r\}$, $|A(p_i^+)| = 2i - 1$ and

$$A(p_1^+) \subseteq A(p_2^+) \subseteq \cdots \subseteq A(p_{2r}^+).$$

It is clear that Algorithm 1 applied to this instance produces, for each $i = 1, \ldots, 2r$, $R_{p_i} = p_i^+$, since $|A(p_i^+)| = 2i - 1 \leq |A(p_i^-)| = 8r - 2i$. Hence, H_R is a complete graph and $\ell(H_R, \mathbf{1}) = 2r$.

However, the optimal solution for this instance is $R^*_{p_i} = p_i^+$, for $i = 1, \ldots, r$ and $R^*_{p_i} = p_i^-$, for $i = r + 1, \ldots, 2r$, with $\ell(H_{R^*}, \mathbf{1}) = r$, which shows that the factor 2 is best possible for this algorithm.

3 A FPT ALGORITHM FOR UNIT WEIGHTS

Our last result is a FPT algorithm for the Load Minimization Problem restricted to cycles for which we still need some additional definitions.

Given a routing R, two parallel demands p and q *collide* in R if $A(R_p) \cup A(R_q)$ is the set of arcs of the cycle. In this case, we call the pair (p, q) a *collision* in R.

Note that the solution R given by the previously introduced 2-approximation algorithm is such that there are no collisions between parallel demands. Now we are going to prove that there is always an optimal solution satisfying this nice, structural property, when all the weights are equal to one.

Theorem 3.1. *The Load Minimization Problem for constant weights at value 1, has an optimal solution without collisions.*

Proof. By contradiction, let us assume that every optimal solution has collisions and take an optimal solution R with the minimum number of them. Let (p, q) be a collision in R such that $|A(R_p)| + |A(R_q)|$ is as large as possible.

Let $o \in \mathcal{D}$ with $e(o) \subseteq V(\overline{R_p})$. Then, $R_o \subseteq \overline{R_p}$. Otherwise, $R_p \subseteq R_o$ and (o, q) would be a collision, because $e(q) \subseteq R_p$. Since $|A(R_p)| < |A(R_o)|$, we get a contradiction. Similarly, if $e(o) \subseteq V(\overline{R_q})$, then $R_o \subseteq \overline{R_q}$.

Consider the new solution R' obtained from R by replacing R_p and R_q by $\overline{R_p}$ and $\overline{R_q}$, respectively.

Clearly, in $H_{R'}$ the demands p and q are not adjacent. Moreover, any neighbor p' of p in $H_{R'}$ is also a neighbor of q in H_R because $R'_p \subseteq R_q$; similarly, any neighbor q' of q in $H_{R'}$ is also a neighbor of p in H_R.

Let C be a set of pairwise adjacent demands in $H_{R'}$. If $\{p, q\} \cap C$ is empty, then the demands in C are also pairwise adjacent in H_R. Whence, $|C| \leq \ell(H_R, \mathbf{1})$.

Since p and q are not adjacent in $H_{R'}$ we can assume that $p \in C$ and $q \notin C$. Hence, the demands in $C \setminus \{p\}$ are adjacent to q in H_R which shows that $|C| \leq \ell(H_R, \mathbf{1})$. Therefore, $\ell(H_{R'}, \mathbf{1}) \leq \ell(H_R, \mathbf{1})$.

It remains to see that the number of collisions in R' is smaller than the ones in R. This follows from the fact that p and q are not involved in any collision in R' and that any collision in R' not involving neither p nor q is a collision in R. □

For a demand $p \in \mathcal{D}$, let $\mathcal{D}(p)$ denote the set of all demands parallel to p. Also, let S^p be the routing of $\mathcal{D}(p) \setminus \{p\}$ given by: $S^p_q \subseteq p^+$, when $e(q) \subseteq V(p^+)$ and $S^p_q \subseteq p^-$, when $e(q) \subseteq V(p^-)$. A demand p is *critical* for a routing R if for all $q \in \mathcal{D}(p) \setminus \{p\}$, $R_q = S^p_q$.

It is easy to see that if R_p is the largest route of a routing R without collisions, then p is critical for R. Notice that the number of routings for which a given demand p is critical is at most $2^{|\mathcal{D} \setminus \mathcal{D}(p)|+1}$.

We can take advantage of this fact to derive a Fixed Parameter Tractable Algorithm for the problem, parameterized by $k = \max\{|\mathcal{D} \setminus \mathcal{D}(p)| : p \in \mathcal{D}\}$.

Theorem 3.2. *The Load Minimization Problem has a FPT algorithm whose time complexity is* $O(2^k|\mathcal{D}|^4)$.

Proof. The algorithm computes, for each demand p, a routing R^p which minimizes $\mathbf{w}(R, \mathbf{1})$, over all routings R of $\mathcal{D}$ for which p is critical for R, *i.e.* $R_q = S_q^p$, for $q \in \mathcal{D}(p) \setminus \{p\}$. The output of the algorithm is a routing R^* that minimizes $\mathbf{w}(R^p)$ over all $p \in \mathcal{D}$.

The algorithm is correct since each routing without collision has a critical demand and then, an optimal solution to our problem without collisions must be R^p, for some $p \in \mathcal{D}$.

For a given routing R, computing $\mathbf{w}(R, \mathbf{1})$ takes time $O(|\mathcal{D}|^3)$. Then, to determine R^p takes time $O(2^k|\mathcal{D}|^3)$, since the set of routings for which p is critical has at most 2^k elements. Therefore, the time complexity of this algorithm is $O(2^k|\mathcal{D}|^4)$. □

Even though we do not known how to find an optimal routing for all the demands we can find an optimal routing for those demands satisfying $\mathcal{D} = \mathcal{D}(p)$.

Proposition 3.3. *Let* $p \in \mathcal{D}$ *be such that* $\mathcal{D} = \mathcal{D}(p)$. *Suppose that* $\mathbf{w}^+ \geq \mathbf{w}^-$, *where* $\mathbf{w}^+ = \mathbf{w}(\{q : S_q^p \subseteq p^+, q \neq p\})$ *and* $\mathbf{w}^- = \mathbf{w}(\{q : S_q^p \subseteq p^-, q \neq p\})$. *Then, there exists an optimal routing* R *without collisions such that* $R_p = p^-$.

Proof. Suppose that R' is an optimal routing without collisions such that $R'_p = p^+$. Then, $\mathbf{w}(R') \geq \mathbf{w}^+ + 1$. Since for each q with $S_q^p \subseteq p^+$, $R'_q = q^+$, to avoid collisions with $R'_p = p^+$. Let R be the extension of S^p to a routing of $\mathcal{D}$ given by $R_p = p^-$ (and $R_q = S_q^p$, for $q \neq p$). Then, we get that $\mathbf{w}(R) = \max\{\mathbf{w}^+, \mathbf{w}^- + 1\} \leq \mathbf{w}^+ + 1$, since $\mathbf{w}^+ \geq \mathbf{w}^-$, by hypothesis, which shows that $\mathbf{w}(R) \leq \mathbf{w}(R')$. Therefore, since S^p has no collisions, R is an optimal routing without collisions. □

ACKNOWLEDGEMENTS

We thank Annegret Wagler for earlier discussions on the Routing and Spectrum Assignment (RSA) problem on trees and cycles, which motivated our work on this topic.

REFERENCES

[1] Steven Cosares and Iraj Saniee. An optimization problem related to balancing loads on sonet rings. Telecommunication Systems, 3(2): 165-–181, 1994.

[2] Chandra Chekuri, Sanjeev Khanna, and F. Bruce Shepherd. An $\sqrt{n}$-approximation for edp in undirected graphs and directed acyclic graphs. Theory of Computing 2, 137—146, 2006.

[3] Julia Chuzhoy, Joseph Naor, New hardness results for congestion minimization and machine scheduling. J. ACM 53(5): 707–721, 2006.

[4] Kostas Christodoulopoulos, Ioannis Tomkos, and Emmanuel Varvarigos, Elastic bandwidth allocation in flexible OFDM-based optical networks, Journal of Lightwave Technology, vol. 29, no. 9, 1354-1366, 2011.

[5] Andras Frank. Edge-disjoint paths in planar graphs. Journal of Combinatorial Theory Series B, 39(2): 164—178, 1985.

[6] Mahmoud Fayez, Iyad Katib, G. N. Rouskas and Hossam Faheem, Spectrum Assignment in Mesh Elastic Optical Networks, 2015 24th International Conference on Computer Communication and Networks (ICCCN), 1–6, 2015. doi: 10.1109/ICCCN.2015.7288470.

[7] Michael Garey and David Johnson, Computers and intractability. Books in the mathematical sciences, Vol. 174. Freeman San Francisco.

[8] W.-L. Hsu, Maximum Weight Clique Algorithms for Circular-Arc Graphs and Circle Graphs, SIAM Journal on Computing, 14:1, 224–231, 1985.

[9] Richard Karp, On the computational complexity of combinatorial problems, Networks 5 45–68, 1975.

[10] Ken-ichi Kawarabayashi, Yusuke Kobayashi and Bruce Reed, The disjoint paths problem in quadratic time, Journal of Combinatorial Theory, Series B 102 424–435, 2012.

[11] Sanjeev Khanna. A polynomial time approximation scheme for the sonet ring loading problem. Bell Labs Technical Journal, 2(2):36—41, 1997.

[12] James Lynch, The equivalence of theorem proving and the interconnection problem, ACM SIGDA Newsletter 5 31—65, 1975.

[13] Asuman Ozdaglar and Dimitri Bertsekas, Routing and wavelength assignment in optical networks, IEEE/ACM Transactions on Networking 11(2), 2003.

[14] Neil Robertson and Paul Seymour, Graph minors. XIII. The disjoint paths problem, Journal of Combinatorial Theory Series B 63 65-–110, 1995.

[15] Shahrzad Shirazipourazad, Chenyang Zhou, Zarah Derakhshandeh and Arunabha Sen, On routing and spectrum allocation in spectrum-sliced optical networks, 2013 Proceedings IEEE INFOCOM 385–389, 2013.

[16] Talebi Sahar, Alam Furqan, Katib Iyad, Khamis, Mohamed, Salama Reda, Rouskas George N, Spectrum management techniques for elastic optical networks: A survey, Optical Switching and Networking 13, 34–48, 2014.

[17] Wang Yang, Cao Xiaojun, Pan Yi, A study of the routing and spectrum allocation in spectrum-sliced elastic optical path networks, 2011 Proceedings IEEE Infocom, 1503–1511, 2011.